Online Appendix for
Multiproduct-Firm Oligopoly: An Aggregative Games Approach

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September 19, 2017

Contents

I The Demand System **4**

I.1 Discrete/Continuous Choice 4

I.2 Representative Consumer Approach 6

II Pricing Game: Preliminaries **12**

II.1 Proof of Lemma A 12

II.2 About the (Log)-Supermodularity of Payoff Functions 14

II.3 About Infinite Prices 15

III About Assumption 1 **16**

III.1 Definitions and Statement of the Theorem 16

III.2 Proof of Theorem I 17

III.3 A Remark on Single-Product Firms 21

III.4 Equilibrium Existence without Assumption 1 23

IV Choke Price **23**

V Equilibrium Uniqueness **26**

V.1 Main Results 26

V.2 Proof of Theorem II 28

V.3 Condition (b) when $\lim_{p_j \to \infty} h_j(p_j) \geq 0$ 35

V.4 Proof of Proposition III 40

V.5 An Index Approach to Equilibrium Uniqueness 42

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VI Functional Forms and Cookbooks for Applied Work **44**
VI.1 Equilibrium Existence: Functional Forms and Cookbook 44
VI.2 Equilibrium Uniqueness: Functional Forms and Cookbook 48

VII Nested Demand Systems and Multi-Stage Discrete / Continuous Choice **50**
VII.1 Multi-Stage Discrete/Continuous Choice 50
VII.2 Representative Consumer Approach 60

VIII Multi-Product Firm Pricing Games and Nested Demand Systems **61**
VIII.1 Definition of the Pricing Game 61
VIII.2 Equilibrium Existence, Uniqueness, and Characterization 62
VIII.3 Discussion 76

IX Additive Aggregation and Demand Systems **79**
IX.1 Characterization Result 79
IX.2 The Generalized Common ι-Markup Property 82

X General Equilibrium **83**
X.1 The Demand System 83
X.2 Multiproduct-Firm Oligopoly Pricing in General Equilibrium 85
X.3 Equilibrium analysis 87
X.4 Special Cases 89

XI Quantity Competition **91**
XI.1 The Demand System 91
XI.2 Assumptions and Technical Preliminaries 91
XI.3 The Quantity-Setting Game and the Firm's Profit-Maximization Problem 92
XI.4 The Additive Constant ι-Markup Property 93
XI.5 Definition and Properties of Output Functions 93
XI.6 Definition and Properties of Markup Fitting-In Functions 94
XI.7 Definition and Properties of Output Fitting-In Functions 95
XI.8 Definition and Properties of the Aggregate Fitting-In Function 96
XI.9 Equilibrium Uniqueness and Sufficiency of First-Order Conditions 97
XI.10 The CES Case 98
XI.11 Firm Scope and Industry Competitiveness under Quantity Competition . 98

XII Firm Scope and the Intensity of Competition **99**

XIII Nested CES and MNL Demands: Type Aggregation and Algorithm **103**
XIII.1 Formal Equivalence between Pricing Games with and without Nests . . . 103
XIII.2 Algorithm 106
XIII.3 Formulas for m' and S' and Preliminary Lemmas 107
XIII.4 Proof of Proposition 6 109

XIV Comparative Statics **111**
XIV.1 Proof of Proposition 3 . . . 111
XIV.2 Proof of Proposition 4 . . . 112
XIV.3 On the Impact of Production Costs on Equilibrium Consumer Surplus . . 112
XIV.4 On the Impact of Production Costs on a Firm's Equilibrium Profit 116

XV Application: Merger Policy **118**
XV.1 Static Merger Analysis . . . 119
XV.2 Dynamic Merger Analysis . . . 122
XV.3 Proofs . . . 124

XVI Application: International Trade **132**
XVI.1 Trade Analysis . . . 133
XVI.2 Formal Statements and Proofs . . . 136

XVIITable of Symbols and Notations **142**

I The Demand System

I.1 Discrete/Continuous Choice

We consider a demand model in which consumers make discrete/continuous choices: Each consumer first decides which product to purchase, and then, how much of this product to consume. This approach captures Novshek and Sonnenschein (1979)'s idea that price-induced demand changes can be decomposed into two effects: An intensive margin effect (consumers purchase less of the product whose price was raised), and an extensive margin effect (some consumers stop purchasing the commodity whose price increased).[1,2]

We formalize discrete/continuous choice as follows. There is a population of consumers with quasi-linear preferences. Each consumer chooses a single product from a finite and non-empty set of products $\mathcal{N} \cup \{0\}$, where good 0 denotes the outside option. After having chosen good $i \in \mathcal{N}$, the consumer under consideration chooses the quantity of that product, and spends the rest of his income on the outside good (or Hicksian composite commodity), the price of which is normalized to one. Conditional on selecting product i, the consumer receives indirect utility $y + v_i(p_i) + \varepsilon_i$, where p_i is the price of product i, y is the consumer's income, and ε_i is a taste shock. By Roy's identity, the consumer purchases $-v_i'(p_i)$ units of good i. We call $-v_i'(p_i)$ the conditional demand for product i. If the consumer chooses the outside option, then he simply receives the utility flow $y + \log H^0 + \varepsilon_0$, where $H^0 \geq 0$. At the product-choice stage, the consumer selects product i only if

$$\forall j \in \mathcal{N}, \quad y + v_i(p_i) + \varepsilon_i \geq y + v_j(p_j) + \varepsilon_j$$

and

$$y + v_i(p_i) + \varepsilon_i \geq y + \log H^0 + \varepsilon_0.$$

We assume that the components of vector $(\varepsilon_j)_{j \in \mathcal{N} \cup \{0\}}$ are identically and independently drawn from a type-1 extreme value distribution. By Holman and Marley's theorem, product i is therefore chosen with probability

$$\begin{aligned}
\mathbb{P}_i(p) &= \Pr\left(v_i(p_i) + \varepsilon_i = \max\left(\log H^0 + \varepsilon_0, \max_{j \in \mathcal{N}}\left(v_j(p_j) + \varepsilon_j\right)\right)\right), \\
&= \frac{\mathrm{e}^{v_i(p_i)}}{\sum_{j \in \mathcal{N}} \mathrm{e}^{v_j(p_j)} + H^0}, \\
&= \frac{h_i(p_i)}{\sum_{j \in \mathcal{N}} h_j(p_j) + H^0},
\end{aligned}$$

[1]Income effects are absent in our quasi-linear world.

[2]See also Hanemann (1984).

where $h_j \equiv \mathrm{e}^{v_j}$ for every j. It follows that the expected demand for product i is given by

$$D_i = \frac{h_i(p_i)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}\left(-v_i'(p_i)\right) = \frac{-h_i'(p_i)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}.$$

In the following, we use the tuple $((h_j)_{j\in\mathcal{N}}, H^0)$ (rather than $(v_j)_{j\in\mathcal{N}}$ and $\log H^0$) as primitives. We assume that all the h functions are $\mathcal{C}^3$ from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$, strictly decreasing, and log-convex. The assumption that h_j is non-increasing and log-convex is necessary and sufficient for v_j to be an indirect subutility function. The assumption that h_j is strictly decreasing means that the demand for product j never vanishes.

To sum up, the demand system generated by the discrete/continuous choice model $((h_j)_{j\in\mathcal{N}}, H^0)$ (when normalizing market size to one) is:

$$D_i\left((p_j)_{j\in\mathcal{N}}\right) = \frac{-h_i'(p_i)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}, \quad \forall i \in \mathcal{N}, \quad \forall (p_j)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}. \tag{i}$$

The conditional demand for good i is $-d\log h_i/dp_i = -h_i'/h_i$. Product i is chosen with probability $h_i/(\sum_j h_j + H^0)$.

The consumer's expected utility can be computed using standard formulas (see, e.g., Anderson, de Palma, and Thisse, 1992):

$$\begin{aligned}\mathbb{E}\left(y + \max\left(\log H^0 + \varepsilon_0, \max_{j\in\mathcal{N}} v_j(p_j) + \varepsilon_j\right)\right) &= y + \log\left(\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)} + H^0\right), \qquad \text{(ii)}\\ &= y + \log\left(\sum_{j\in\mathcal{N}} h_j(p_j) + H^0\right).\end{aligned}$$

Consumer heterogeneity. While the discrete/continuous consumer choice model allows for some type of consumer heterogeneity (different consumers receive different taste shocks and may therefore select different products), it does have the property that all consumers who select the same product choose to purchase the same quantity. However, the model can easily be adapted to accommodate consumer heterogeneity in the quantity purchased of the same product. In particular, suppose that the indirect subutility derived from choosing product j is $v_j(p_j, t_j)$, where $t_j \in \mathbb{R}$ is the consumer's "type" for product j, drawn from the probability distribution $G_j(\cdot)$. The realized value of t_j is observed by the consumer only *after* he has chosen product j. Let $v_j(p_j) = \int v_j(p_j, t_j)dG_j(t_j)$ be the expected indirect utility derived from product j. Then, product i is chosen with probability $\exp v_i(p_i)/(\sum_j \exp v_j(p_j) + H^0)$. Under some technical conditions (which allow us to differentiate under the integral sign), the consumer's expected conditional demand for product j is:

$$\int -\frac{\partial}{\partial p_j} v_j(p_j, t_j)dG_j(t_j) = -\frac{\partial}{\partial p_j}\int v_j(p_j, t_j)dG_j(t_j) = -v_j'(p_j).$$

Therefore, if we define $h_j(p_j) = \exp(v_j(p_j))$ for every j, then the expected (unconditional) demand for product i is still given by equation (i). Differentiating once more under the integral sign, we also see that $v_j(\cdot)$ is decreasing and convex if $v_j(\cdot, t_j)$ is decreasing and convex for every t_j. Therefore, discrete/continuous choice with consumer heterogeneity gives rise to the same class of demand systems as discrete/continuous choice without heterogeneity.

Note however that, if the consumer observes his vector of types *before* choosing a variety, then the implied demand system becomes a mixture of equation (i). We are not able to handle such mixtures of demand systems, because they no longer give rise to an aggregative game. This implies in particular that our approach cannot accommodate random coefficient logit demand systems. At the end of Section VII.1, we show how a restricted class of random coefficient logit demand systems can be handled.

I.2 Representative Consumer Approach

We now show that the demand system (i) can also be derived from the maximization of the utility function of a representative consumer with quasi-linear preferences. To this end, we first prove the following proposition:

Proposition I. *Let $\mathcal{N}$ be a finite and non-empty set. For every $k \in \mathcal{N}$, let h_k (resp. g_k) be a $\mathcal{C}^2$ (resp. $\mathcal{C}^1$) function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Suppose that $h'_k < 0$ for every k. Define the demand system D as follows:*

$$D_k\left((p_j)_{j\in\mathcal{N}}\right) = \frac{g_k(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall \left(p_j\right)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$$

The following assertions are equivalent:[3]

(i) D is quasi-linearly integrable.

(ii) There exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$. Moreover, $h''_k > 0$ for every $k \in \mathcal{N}$, and $\sum_{k\in\mathcal{N}} \gamma_k \leq \sum_{k\in\mathcal{N}} h_k$, where $\gamma_k = h'^2_k / h''_k$ for every $k \in \mathcal{N}$.

When this is the case, the function $v(.)$ is an indirect subutility function for the associated demand system if and only if there exists $\beta \in \mathbb{R}$ such that $v(p) = \alpha \log\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right) + \beta$ for every $p >> 0$.

To prove Theorem I, we first state and prove two technical lemmas:

[3] Quasi-linear integrability and indirect subutility functions are defined in Nocke and Schutz (2017), Definitions 3 and 4.

Lemma I. *For every $n \geq 1$, for every $(\alpha_i)_{1\leq i\leq n} \in \mathbb{R}^n$, define*

$$\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right) = \begin{pmatrix} 1-\alpha_1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{pmatrix}$$

Then,[4]

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-1)^n \left(\left(\prod_{k=1}^{n} \alpha_k\right) - \sum_{j=1}^{n}\left(\prod_{\substack{1\leq k\leq n \\ k\neq j}} \alpha_k\right)\right)$$

Moreover, the matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is negative semi-definite if and only if $\alpha_i \geq 1$ for all $1 \leq i \leq n$ and

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \leq 1.$$

Proof. We prove the first part of the lemma by induction on $n \geq 1$. Start with $n = 1$. Then,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = 1 - \alpha_1 = (-1)^1(\alpha_1 - 1),$$

so the property is true for $n = 1$.

Next, let $n \geq 2$, and assume the property holds for all $1 \leq m < n$. By n-linearity of the determinant,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-\alpha_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix}.$$

Applying Laplace's formula to the first column, we can see that the first determinant is, in fact, equal to $\det\left(\mathcal{M}\left((\alpha_i)_{2\leq i\leq n}\right)\right)$. The second determinant can be simplified by using n-linearity one more time:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix} = -\alpha_2 \begin{vmatrix} 1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix},$$

$$= -\alpha_2 \det\left(\mathcal{M}\left(0, (\alpha_i)_{3\leq i\leq n}\right)\right) + 0,$$

[4] We adopt the convention that the product of an empty collection of real numbers is equal to 1.

where the second line follows again from Laplace's formula and from the fact that the first two rows of the second matrix in the first line's right-hand side are collinear. Therefore,

$$\begin{aligned}
\det \mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right) =& -\alpha_1 \det\left(\mathcal{M}\left((\alpha_i)_{2\leq i\leq n}\right)\right) - \alpha_2 \det\left(\mathcal{M}\left(0,(\alpha_i)_{3\leq i\leq n}\right)\right), \\
=& -\alpha_1(-1)^{n-1}\left(\left(\prod_{k=2}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{2\leq k\leq n\\ k\neq j}}\alpha_k\right)\right) \\
& -\alpha_2(-1)^{n-1}\left(0-\prod_{k=3}^{n}\alpha_k\right), \\
=& (-1)^n\left(\left(\prod_{k=1}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{1\leq k\leq n\\ k\neq j}}\alpha_k\right) - \prod_{k=2}^{n}\alpha_k\right), \\
=& (-1)^n\left(\left(\prod_{k=1}^{n}\alpha_k\right) - \sum_{j=1}^{n}\left(\prod_{\substack{1\leq k\leq n\\ k\neq j}}\alpha_k\right)\right).
\end{aligned}$$

We now turn our attention to the second part of the lemma. Assume first that the matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is negative semi-definite. Then, all its diagonal terms have to be non-positive, i.e., $\alpha_i \geq 1$ for all i. Besides, the determinant of this matrix should be non-negative (resp. non-positive) if n is even (resp. odd). Put differently, the sign of the determinant should be $(-1)^n$ or 0. Since the α's are all different from zero, this determinant can be simplified as follows:

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-1)^n\left(\prod_{k=1}^{n}\alpha_k\right)\left(1-\sum_{k=1}^{n}\frac{1}{\alpha_k}\right).$$

This expression has sign $(-1)^n$ or 0 if and only if $\sum_{k=1}^{n}\frac{1}{\alpha_k} \leq 1$.

Conversely, assume that the α's are all ≥ 1, and that $\sum_{k=1}^{n}\frac{1}{\alpha_k} \leq 1$. The characteristic polynomial of the matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is defined as

$$P(X) = \begin{vmatrix} 1-\alpha_1-X & 1 & \cdots & 1 \\ 1 & 1-\alpha_2-X & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n-X \end{vmatrix}.$$

This determinant can be calculated using the first part of the lemma. For every $X > 0$,

$$(-1)^n P(X) = \underbrace{\left(\prod_{k=1}^{n}(\alpha_k+X)\right)}_{>0}\left(1-\sum_{k=1}^{n}\frac{1}{\alpha_k+X}\right),$$

$$> \underbrace{\left(\prod_{k=1}^{n}(\alpha_k + X)\right)}_{>0} \underbrace{\left(1 - \sum_{k=1}^{n}\frac{1}{\alpha_k}\right)}_{\geq 0},$$
$$> 0.$$

Therefore, $P(X)$ has no strictly positive root, the matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ has no strictly positive eigenvalue, and this matrix is therefore negative semi-definite. □

Lemma II. *Let M be a symmetric n-by-n matrix, $\lambda \neq 0$, and $1 \leq k \leq n$. Let A^k be the matrix obtained by dividing the k-th line and the k-th column of M by λ. Then, M is negative semi-definite if and only if A^k is negative semi-definite.*

Proof. Suppose M is negative semi-definite, and let $X \in \mathbb{R}^n$. Write A^k as $(a_{ij})_{1\leq i,j\leq n}$ and M as $(m_{ij})_{1\leq i,j\leq n}$. Finally, define Y as the n-dimensional vector obtained by dividing X's k-th component by λ. Then,

$$\begin{aligned}
X'A^kX &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j,\\
&= \left(\sum_{\substack{1\leq i\leq n\\ i\neq k}}\sum_{\substack{1\leq j\leq n\\ j\neq k}} a_{ij}x_ix_j\right) + 2x_k\sum_{\substack{1\leq i\leq n\\ i\neq k}} a_{ik}x_i + x_k^2a_{kk},\\
&= \left(\sum_{\substack{1\leq i\leq n\\ i\neq k}}\sum_{\substack{1\leq j\leq n\\ j\neq k}} m_{ij}x_ix_j\right) + 2\frac{x_k}{\lambda}\sum_{\substack{1\leq i\leq n\\ i\neq k}} m_{ik}x_i + \left(\frac{x_k}{\lambda}\right)^2 m_{kk},\\
&= \left(\sum_{\substack{1\leq i\leq n\\ i\neq k}}\sum_{\substack{1\leq j\leq n\\ j\neq k}} m_{ij}y_iy_j\right) + 2y_k\sum_{\substack{1\leq i\leq n\\ i\neq k}} m_{ik}y_i + y^2m_{kk},\\
&= Y'MY,\\
&\leq 0, \text{ since } M \text{ is negative semi-definite.}
\end{aligned}$$

Therefore, A^k is negative semi-definite.

The other direction is now immediate, since M can be obtained by dividing the k-th line and the k-th column of the matrix A^k by $1/\lambda$. □

We can now prove Proposition I:

Proof. To simplify notation, assume without loss of generality that $\mathcal{N} = \{1, \ldots, n\}$. For every $p >> 0$, put $J(p) = \left(\frac{\partial D_i}{\partial p_j}(p)\right)_{1\leq i,j\leq n}$. Theorem 1 in Nocke and Schutz (2017) states that D is quasi-linearly integrable if and only if $J(p)$ is symmetric and negative semi-definite for every $p >> 0$.

We first show that the matrix $J(p)$ is symmetric for every p if and only if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$. If $J(p)$ is symmetric for every p, then, for every $1 \leq i, j \leq n$ such that $i \neq j$, for every $p >> 0$,

$$-\frac{h'_j(p_j)g_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = J_{i,j}(p) = J_{j,i}(p) = -\frac{h'_i(p_i)g_j(p_j)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2}.$$

It follows that, for every $1 \leq i \leq n$, for every $x > 0$,

$$\frac{h'_i(x)}{g_i(x)} = \frac{h'_1(1)}{g_1(1)} \equiv -\beta \tag{iii}$$

If $\beta = 0$, then $h'_i = 0$ for every i, which violates the assumption that h_i is strictly decreasing. Therefore, $\beta \neq 0$, and we can define $\alpha \equiv 1/\beta$. It follows that $g_i = -\alpha h'_i$. Since $g_i > 0$ and $h'_i \leq 0$, we can conclude that $\alpha > 0$. Conversely, if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$, then, for every $1 \leq i, j \leq n$, $i \neq j$, for every $p >> 0$,

$$J_{i,j}(p) = -\frac{h'_j(p_j)g_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = \alpha\frac{h'_j(p_j)h'_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = J_{j,i}(p),$$

and the matrix $J(p)$ is therefore symmetric for every p.

Next, suppose that there exists $\alpha > 0$ such that, for every $1 \leq k \leq n$, $g_k = -\alpha h'_k$. We want to show that $J(p)$ is negative semi-definite for every $p >> 0$ if and only if $h''_k > 0$ for every $1 \leq k \leq n$, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Fix $p >> 0$. To ease notation, we write $h_k = h_k(p_k)$ for every k, and define $H \equiv \sum_{k\in\mathcal{N}} h_k$. We obtain the following expression for the matrix $J(p)$:

$$J(p) = \frac{\alpha}{H^2}\begin{pmatrix} (h'_1)^2 - h''_1 H & h'_1 h'_2 & \cdots & h'_1 h'_n \\ h'_2 h'_1 & (h'_2)^2 - h''_2 H & \cdots & h'_2 h'_n \\ \vdots & \vdots & \ddots & \vdots \\ h'_n h'_1 & h'_n h'_2 & \cdots & (h'_n)^2 - h''_n H \end{pmatrix}.$$

$J(p)$ is negative semi-definite if and only if

$$\begin{pmatrix} (h'_1)^2 - h''_1 H & h'_1 h'_2 & \cdots & h'_1 h'_n \\ h'_2 h'_1 & (h'_2)^2 - h''_2 H & \cdots & h'_2 h'_n \\ \vdots & \vdots & \ddots & \vdots \\ h'_n h'_1 & h'_n h'_2 & \cdots & (h'_n)^2 - h''_n H \end{pmatrix}$$

is negative semi-definite. Applying Lemma II n times (by dividing row k and column k by

h'_k, $1 \leq k \leq n$), this is equivalent to the matrix

$$\begin{pmatrix} 1 - \frac{h''_1}{\left(h'_1\right)^2} H & 1 & \cdots & 1 \\ 1 & 1 - \frac{h''_2}{\left(h'_2\right)^2} H & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \frac{h''_n}{(h'_n)^2} H \end{pmatrix}$$

being negative semi-definite. By Lemma I, this holds if and only if $\frac{h''_k}{\left(h'_k\right)^2} H \geq 1$ for all $1 \leq k \leq n$, and $\frac{1}{H} \sum_{k=1}^n \frac{\left(h'_k\right)^2}{h''_k} \leq 1$. This is equivalent to $h''_k > 0$ for all k, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Finally, Nocke and Schutz (2017) show that, v is an indirect subutility function for the demand system D if and only if $\nabla v = -D$. Clearly, this is equivalent to

$$v(p) = \alpha \log \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right) + \beta, \quad \forall p >> 0,$$

where $\beta \in \mathbb{R}$ is a constant of integration. □

Proposition I immediately implies the following corollary:

Corollary I. *Let D be the demand system generated by the discrete/continuous choice model $((h_j)_{j \in \mathcal{N}}, H^0)$. D is quasi-linearly integrable. Moreover, v is an indirect subutility function for D if and only if there exists a constant $\alpha \in \mathbb{R}$ such that $v((p_j)_{j \in \mathcal{N}}) = \alpha + \log \left(\sum_{j \in \mathcal{N}} h_j(p_j) + H^0 \right)$.*

Proof. Note that, for every product i,

$$(\log h_i)'' = \frac{h''_i h_i - h'^2_i}{h_i^2}.$$

By log-convexity of h_i, $h''_i > 0$. Moreover,

$$(\log h_i)'' = \frac{h''_i}{h_i^2} (h_i - \gamma_i) \geq 0.$$

Hence, $h_i \geq \gamma_i$ for every i. This implies in particular that

$$\sum_{k \in \mathcal{N}} h_k + H^0 \geq \sum_{k \in \mathcal{N}} \gamma_k.$$

For every $i \in \mathcal{N}$, let $\widetilde{h}_i = h_i + H^0/|\mathcal{N}|$. Note that, for every i and p,

$$\widetilde{D}_i(p) \equiv \frac{-\widetilde{h}_i'(p_i)}{\sum_{j\in\mathcal{N}} \widetilde{h}_j(p_j)} = \frac{-h_i'(p_i)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0} = D_i(p).$$

Clearly, $(\widetilde{h}_j)_{j\in\mathcal{N}}$ satisfies condition (ii) in Proposition I. Hence, the demand system $\widetilde{D} = D$ is quasi-linearly integrable. Moreover, v is an indirect subutility function for that demand system if and only if

$$v(p) = \alpha + \log \sum_{j\in\mathcal{N}} \widetilde{h}_j(p_j) = \alpha + \log\left(\sum_{j\in\mathcal{N}} h_j(p_j) + H^0\right)$$

for some $\alpha \in \mathbb{R}$. □

Hence, any demand system that can be derived from discrete/continuous choice can also be derived from quasi-linear utility maximization. The second part of the corollary says that the expected utility of a consumer engaging in discrete/continuous choice and the indirect utility of the associated representative consumer coincide (up to an additive constant). The results we derive on consumer welfare therefore do not depend on the way the demand system has been generated. Whether we use discrete/continuous choice or a representative consumer approach, all that matters is the value of the aggregator H.

II Pricing Game: Preliminaries

II.1 Proof of Lemma A

Proof. **(a)** We first show that $\lim_{p\to\infty} ph'(p)$ exists. By the fundamental theorem of calculus, for every $p > 0$,

$$h(p) = h(1) + \int_1^p h'(x)dx = h(1) + ph'(p) - h'(1) - \int_1^p xh''(x)dx,$$

where the second line was obtained by integrating by parts. Therefore, $ph'(p) = h(p) - h(1) + h'(1) + \int_1^p xh''(x)dx$. Since h is positive and decreasing, that function has a finite limit at ∞. We now show that $\int_1^p xh''(x)dx$ also has a limit at infinity. Since h is log-convex, $(\log h)'' = \frac{h''h - h'^2}{h^2} \geq 0$. It follows that $h'' \geq 0$. Therefore, the function $p \mapsto \int_1^p xh''(x)dx$ is non-decreasing, and that function has a limit at infinity. It follows that $\lim_{p\to\infty} ph'(p)$ exists. Since $h' < 0$, that limit is non-positive.

Assume for a contradiction that $\lim_{p\to\infty} ph'(p) < 0$. Then, there exist $\varepsilon_0 > 0$ and $p_0 > 0$ such that $ph'(p) \leq -\varepsilon_0$ for all $p \geq p_0$. Rewrite this inequality as $h'(p) \leq -\varepsilon_0/p$, and integrate it between p^0 and p to get

$$h(p) - h(p_0) \leq -\varepsilon_0 \log\left(\frac{p}{p_0}\right) \underset{p\to\infty}{\longrightarrow} -\infty.$$

Therefore, $\lim_{p\to\infty} h(p) = -\infty$. This contradicts the assumption that $h > 0$.

Therefore, $\lim_{p\to\infty} ph'(p) = 0$, and $\lim_{p\to\infty} h'(p) = 0$.

(b) Assume for a contradiction that $\iota(p) \leq 1$ for all $p > 0$. Then, for all $p > 0$, $ph''(p) + h'(p) \leq 0$, i.e., $\frac{d}{dp}(ph'(p)) \leq 0$. It follows that $ph'(p) \leq h'(1)$ for all $p \geq 1$. Taking the limit as p goes to infinity and using point (a), we obtain that $h'(1) \geq 0$, a contradiction.

Therefore, there exists $\hat{p} > 0$ such that $\iota(\hat{p}) > 1$. It follows that

$$\underline{p} \equiv \inf\{p \in \mathbb{R}_{++} : \quad \iota(p) > 1\} < \infty.$$

We prove two claims:

Claim 1: $\underline{p} \notin \{p > 0 : \ \iota(p) > 1\}$.

If $\underline{p} = 0$, then this is obvious. If instead $\underline{p} > 0$, then the claim follows from the continuity of ι.

Claim 2: $\iota(y) \geq \iota(x)$ whenever $0 < x < y$ and $\iota(x) > 1$.

Assume for a contradiction that $\iota(y) < \iota(x)$. Put $S = \{z \in [x, y] : \ \iota(z) \leq 1\}$. If S is empty, then $\iota(z) > 1$ for every $z \in [x, y]$. Hence, $\iota'(z) \geq 0$ for every $z \in [x, y]$. It follows that $\iota(y) \geq \iota(x)$, which is a contradiction.

Next, assume that S is not empty. Then, $\hat{y} \equiv \inf S \in [x, y]$. Moreover, by continuity of ι, and since $\iota(x) > 1$, $\iota(\hat{y}) = 1$. In addition, $\iota(z) > 1$ for every $z \in [x, \hat{y})$. Using the same reasoning as above, it follows that

$$1 = \iota_k(\hat{y}) \geq \iota_k(x) > 1,$$

which is a contradiction.

Combining Claims 1 and 2, it follows that $\{p > 0 : \ \iota(p) > 1\} = \big(\underline{p}, \infty\big)$, and that ι is non-decreasing on $\big(\underline{p}, \infty\big)$, which proves point (b).

(c) Since ι is non-decreasing and strictly greater than 1 on $(\underline{p}, \infty)$, $\bar{\mu}$ exists, and is strictly greater than 1.

(d) Let $p > \underline{p}$. Note that

$$\gamma(p) = \frac{-h'(p)}{ph''(p)}\left(p(-h'(p))\right) = \frac{-ph'(p)}{\iota(p)}.$$

Therefore,

$$\gamma'(p) = \frac{1}{(\iota(p))^2}\left(-\left(ph''(p)+h'(p)\right)\times\iota(p)+\iota'(p)\times ph'(p)\right),$$
$$= \frac{1}{(\iota(p))^2}\left(-h'(p)\left(1-\iota(p)\right)\iota(p)+\iota'(p)ph'(p)\right) < 0,$$

as $\iota' \geq 0$ and $\iota(p) > 1$ for all $p > \underline{p}$.

(e) The result follows immediately from the fact that $\gamma(p) = -ph'(p)/\iota(p)$ (see above), $\lim_{p\to\infty} ph'(p) = 0$ (point (a)), and $\lim_{\infty}\iota > 0$ (point (c)).

(f) Suppose $\bar{\mu} < \infty$ and $\lim_{p\to\infty} h(p) = 0$. For all $p > \underline{p}$,

$$\rho(p) = \frac{h(p)h''(p)}{(h'(p))^2} = \frac{ph''(p)}{-h'(p)}\frac{h(p)}{-ph'(p)} = \iota(p)\frac{h(p)}{-ph'(p)}.$$

By assumption, $\lim_{p\to\infty} h(p) = 0$. By point (a), $\lim_{p\to\infty} -ph'(p) = 0$. Moreover,

$$\lim_{p\to\infty}\frac{\frac{d}{dp}h(p)}{\frac{d}{dp}\left(-ph'(p)\right)} = \lim_{p\to\infty}\frac{h'(p)}{-h'(p)-ph''(p)} = \lim_{p\to\infty}\frac{1}{\iota(p)-1} = \frac{1}{\bar{\mu}-1}.$$

Therefore, by L'Hospital's rule, $\lim_{p\to\infty}\frac{h(p)}{-ph'(p)} = \frac{1}{\bar{\mu}-1}$, and $\lim_{p\to\infty}\rho(p) = \frac{\bar{\mu}}{\bar{\mu}-1}$. □

II.2 About the (Log)-Supermodularity of Payoff Functions

Fix a pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ satisfying Assumption 1, and let $f \in \mathcal{F}$ such that $|f| \geq 2$. Fix a vector of prices for firm f's rivals $(p_j)_{j\in\mathcal{N}\setminus f}$, and let $H^{0\prime} = \sum_{j\notin f} h_j(p_j) + H^0$. We introduce the following notation: $\nu_i(p_i) = \frac{p_i - c_i}{p_i}\iota_i(p_i)$ for every i and $p_i > 0$.

We first show that Π^f is neither supermodular nor submodular in $(p_j)_{j\in f}$. Let $i \neq k$ in f.

$$\frac{\partial^2\Pi^f}{\partial p_i\partial p_k} = \frac{\partial}{\partial p_k}\left(\frac{-h_i'(p_i)}{H}\left(1-\nu_i(p_i)+\Pi^f(p)\right)\right),$$
$$= -h_i'\left(\frac{-h_k'}{H^2}(1-\nu_i+\Pi^f)+\frac{1}{H}\frac{-h_k'}{H}\left(1-\nu_k+\Pi^f\right)\right),$$
$$= \frac{h_i'h_k'}{H^2}\left((1-\nu_i+\Pi^f)+(1-\nu_k+\Pi^f)\right), \tag{iv}$$

where we have used the expression of marginal profit derived in equation (13).

Assume in addition that firm f's profile of prices satisfies the constant ι-markup property.

Then, equation (iv) can be simplified as follows (see the end of the proof of Lemma F):

$$\frac{\partial^2 \Pi^f}{\partial p_i \partial p_k} = \frac{2h_i' h_k'}{H^2}\left(1 - \mu^f + \frac{1}{H}\mu^f \sum_{j\in f} \gamma_j(r_j(\mu^f))\right),$$

$$= -\frac{2h_i' h_k'}{H^3}\underbrace{\left((\mu^f - 1)\left(H^{0\prime} + \sum_{j\in f} h_j(r_j(\mu^f))\right) - \mu^f \sum_{j\in f}\gamma_j(r_j(\mu^f))\right)}_{\equiv \phi(\mu^f)}.$$

We have shown in the proof of Lemma G that $\phi(\mu^f)$ is strictly positive when μ^f is large, and strictly negative when μ^f is small. It follows that Π^f is neither supermodular nor submodular in $(p_j)_{j\in f}$.

Next, we show that Π^f is neither log-supermodular nor log-submodular in $(p_j)_{j\in f}$. Let $i \neq k$ in f.

$$\frac{\partial^2 \log \Pi^f}{\partial p_i \partial p_k} = \frac{\partial}{\partial p_k}\left(\frac{-h_i' - (p_i - c_i)h_i''}{\sum_{j\in f}(p_j - c_j)(-h_j')} + \frac{-h_i'}{H}\right),$$

$$= -\frac{\left(-h_i' - (p_i - c_i)h_i''\right)\left(-h_k' - (p_k - c_k)h_k''\right)}{\left(\sum_{j\in f}(p_j - c_j)(-h_j')\right)^2} + \frac{h_i' h_k'}{H^2},$$

$$= \frac{h_i' h_k'}{H^2}\left(1 - \frac{(\nu_i - 1)(\nu_k - 1)}{(\Pi^f)^2}\right).$$

Again, if firm f's profile of prices has the constant ι-markup property, then

$$\frac{\partial^2 \Pi^f}{\partial p_i \partial p_k} = \frac{h_i' h_k'}{H^2}\left(1 - \left(\frac{\mu^f - 1}{\Pi^f}\right)^2\right).$$

Note that

$$\frac{\mu^f - 1}{\Pi^f} = 1 + \frac{\phi(\mu^f)}{\mu^f \sum_{j\in f}\gamma_j(r_j(\mu^f))}.$$

Let μ^{f*} be the unique solution of equation $\phi(\mu^f) = 0$. Then, by continuity, for μ^f close enough to μ^{f*} and strictly below μ^{f*}, $(\mu^f - 1)/\Pi^f \in (0,1)$, and, therefore, $\partial^2\Pi^f/\partial p_i \partial p_k > 0$. For μ^f close enough to μ^{f*} and strictly above μ^{f*}, $(\mu^f - 1)/\Pi^f > 1$, and, therefore, $\partial^2\Pi^f/\partial p_i \partial p_k < 0$. Therefore, Π^f is neither log-supermodular nor log-submodular in $(p_j)_{j\in f}$.

II.3 About Infinite Prices

We first argue that the idea that product k is simply not supplied when $p_k = \infty$ is consistent with the discrete/continuous choice interpretation of the demand system. In the discrete/continuous choice model, a consumer receives a type-1 extreme value draw ε_k for product k even when $p_k = \infty$. Three cases can arise when the price is infinite: (i) The conditional demand is positive ($\lim_{p_k \to \infty} -h'_k(p_k)/h_k(p_k) > 0$), in which case the choice probability must be equal to zero ($\lim_{p_k \to \infty} h_k(p_k) = 0$). (ii) The choice probability is positive ($\lim_{p_k \to \infty} h_k(p_k) > 0$), in which case the conditional demand must be equal to zero ($\lim_{p_k \to \infty} -h'_k(p_k)/h_k(p_k) = 0$). (iii) Both the conditional demand and the choice probability are equal to zero.[5] In all three cases, the consumer does not consume a positive quantity of the good when the price is infinite, which is consistent with the interpretation that the product is simply not available.

An alternative way of allowing for infinite prices would be to define the profit function for finite prices first, and then extend it by continuity to price vectors that have infinite components. In the proof of Lemma C in the paper, we show that, if the price vector $\hat{p} \in (0, \infty]^{\mathcal{N}}$ has a least one finite component, then $\lim_{p \to \hat{p}} \Pi^f(p)$ coincides with the value of $\Pi^f(\hat{p})$ defined in equation (2). There is, however, an important exception. If $p_j = \infty$ for every j, then $\lim_{p \to \hat{p}} \Pi^f(p)$ does not necessarily exist. For instance, with CES or MNL demands, firms' profits do not have a limit when all prices go to infinity.

III About Assumption 1

In this section, we formalize and prove our statement that Assumption 1 is the weakest assumption under which an approach based on first-order conditions is valid. We also show how to prove equilibrium existence without Assumption 1.

III.1 Definitions and Statement of the Theorem

In the following, we denote by $\mathcal{H}$ the set of $\mathcal{C}^3$, strictly decreasing and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. $\mathcal{H}^\iota$ is the set of functions in $\mathcal{H}$ that satisfy Assumption 1.

We first define a multiproduct firm as a collection of products, along with a constant unit cost for each product:

Definition 1. *A multiproduct firm is a pair $((h_j)_{j \in \mathcal{N}}, (c_j)_{j \in \mathcal{N}})$, where $\mathcal{N} = \{1, \ldots, n\}$ is a finite and non-empty set, and for every $j \in \mathcal{N}$, $h_j \in \mathcal{H}$, and $c_j > 0$. The profit function associated with multi-product firm M is:*

$$\Pi(M)\left(p, H^0\right) = \sum_{k \in \mathcal{N}} (p_k - c_k) \frac{-h'_k(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j) + H^0}, \quad \forall p \in \mathbb{R}^{\mathcal{N}}_{++}, \ \forall H^0 > 0.$$

[5] To see this, suppose that $\lim_{p \to \infty} -h'(p)/h(p) = l > 0$ (the limit exists, since h is log-convex), where we have dropped the product subscript to ease notation. There exists $p_0 > 0$ such that $-h'(p)/h(p) > l/2$ for all $p \geq p_0$. Integrating this inequality, we see that $-\log\left(\frac{h(p)}{h(p_0)}\right) > \frac{l}{2}(p - p_0)$ for all $p > p_0$. Taking exponentials on both side, and letting p go to infinity, we obtain that $\lim_{p \to \infty} h(p) = 0$. Conversely, $\lim_{p \to \infty} h(p) > 0$ implies that $\lim_{p \to \infty} -h'(p)/h(p) = 0$.

As in the paper, H^0 represents the value of the outside option. Our goal is to derive conditions under which the profit function $\Pi(M)(\cdot, H^0)$ is well-behaved.

In the following, it will be useful to study multiproduct firms that can be constructed from a set of products (i.e., a set of indirect subutility functions) smaller than $\mathcal{H}$:

Definition 2. *The set of multiproduct firms that can be constructed from the set $\mathcal{H}' \subseteq \mathcal{H}$ is:*

$$\mathcal{M}(\mathcal{H}') = \bigcup_{n \in \mathbb{N}_{++}} \left(\mathcal{H}'^n \times \mathbb{R}^n_{++}\right).$$

We can now define well-behaved multiproduct firms and well-behaved sets of products:

Definition 3. *We say that multiproduct firm $M \in \mathcal{M}(\mathcal{H})$ is well-behaved if for every $(p, H^0) \in \mathbb{R}^{n+1}_{++}$, $\nabla_p \Pi(M)(p, H^0) = 0$ implies that p is a local maximizer of $\Pi(M)(., H^0)$. We say that the product set $\mathcal{H}' \subseteq \mathcal{H}$ is well-behaved if every $M \in \mathcal{M}(\mathcal{H}')$ is well-behaved.*

Put differently, a set of products is well-behaved if for every multiproduct firm that can be constructed from this set, for every value the outside option H^0 can take, first-order conditions are sufficient for local optimality. In the following, we look for the "largest" well-behaved set of products, where the meaning of "large" will be made more precise shortly.

We define the set of CES products as follows:

$$\mathcal{H}^{CES} = \left\{h \in \mathcal{H} : \quad \exists (a, \sigma) \in \mathbb{R}_{++} \times (1, \infty) \text{ s.t. } \forall p > 0,\ h(p) = ap^{1-\sigma}\right\}.$$

We have shown in the paper that $\mathcal{H}^{CES} \subseteq \mathcal{H}^{\iota}$.

We are now in a position to state our theorem:

Theorem I. *$\mathcal{H}^{\iota}$ is the largest (in the sense of set inclusion) set $\mathcal{H}' \subseteq \mathcal{H}$ such that $\mathcal{H}^{CES} \subseteq \mathcal{H}'$ and $\mathcal{H}'$ is well-behaved.*

In words, $\mathcal{H}^{\iota}$ is the largest set of products that contains CES products and that is well-behaved. Rephrasing this result in terms of pricing games, this means that pricing games based on sets of products larger than $\mathcal{H}^{\iota}$ are not well-behaved, and that an aggregative games approach based on first-order conditions is not valid.

III.2 Proof of Theorem I

We first make the dependence of the function ν_k (which maps prices into ι-markups) on the marginal cost c_k explicit by writing $\nu_k(p_k, c_k) \equiv \frac{p_k - c_k}{p_k} \iota_k(p_k)$. Note that

$$\frac{\partial \nu_k}{\partial p_k} = \frac{c_k}{p_k^2} \iota_k(p_k) + \frac{p_k - c_k}{p_k} \iota_k'(p_k). \tag{v}$$

In addition, since $\iota_k(p_k) = p_k \frac{-h_k'(p_k)}{\gamma_k(p_k)}$, we also have that

$$\frac{\partial \nu_k}{\partial p_k} = \frac{(\nu_k(p_k, c_k) - 1)\, h_k'(p_k) - \nu_k(p_k, c_k)\gamma_k'(p_k)}{\gamma_k(p_k)}. \tag{vi}$$

Differentiating the monopolist's profit with respect to p_k, we obtain:

$$\begin{aligned} \frac{\partial \Pi\,(M)}{\partial p_k} &= \frac{-h_k'(p_k)}{H}\left(1 - \frac{p_k - c_k}{p_k} p_k \frac{-h_k''(p_k)}{-h_k'(p_k)} + \sum_{j \in \mathcal{N}} (p_j - c_j) \frac{-h_j'(p_j)}{H}\right), \\ &= \frac{-h_k'(p_k)}{H}\left(1 - \nu_k(p_k, c_k) + \sum_{j \in \mathcal{N}} \nu_j(p_j, c_j) \frac{\gamma_j(p_j)}{H}\right), \end{aligned} \tag{vii}$$

where $H = \sum_{j \in \mathcal{N}} h_j(p_j) + H^0$. Therefore, if the first-order conditions hold at price vector p, then, for every k in $\mathcal{N}$,

$$\nu_k(p_k, c_k) = 1 + \sum_{j \in \mathcal{N}} \nu_j(p_j, c_j) \frac{\gamma_j(p_j)}{H}. \tag{viii}$$

Since the right-hand side of the above equation does not depend on the identity of product k, it follows that p satisfies the common-ι markup property:

$$\nu(p_i, c_i) = \nu(p_j, c_j), \quad \forall i, j \in \mathcal{N}.$$

This allows us to rewrite the first-order condition for product k as follows:

$$\nu_k(p_k, c_k)\left(1 - \sum_{j \in \mathcal{N}} \frac{\gamma_j(p_j)}{H}\right) = 1. \tag{ix}$$

Since we are interested in the sufficiency of first-order conditions for local optimality, we need to calculate the Hessian of the monopolist's profit function. This is done in the following lemma:

Lemma III. *Let $M \in \mathcal{M}(\mathcal{H})$, $p >> 0$ and $H^0 > 0$. If $\nabla_p \Pi(M)\,(p, H^0) = 0$, then the Hessian of $\Pi(M)\,(., H^0)$, evaluated at price vector p, is diagonal, with typical diagonal element*

$$\frac{h_k'(p_k)}{H^0 + \sum_{j \in \mathcal{N}} h_j(p_j)} \frac{\partial \nu_k}{\partial p_k}(p_k, c_k).$$

Proof. Let $M = \left((h_j)_{j \in \mathcal{N}}, (c_j)_{j \in \mathcal{N}}\right) \in \mathcal{M}(\mathcal{H})$. Let $p >> 0$ and $H^0 > 0$, and suppose that $\nabla_p \Pi(M)(p, H^0) = 0$. For every $1 \leq k \leq n$,

$$\frac{\partial^2 \Pi(M)}{\partial p_k^2} = \frac{-h_k'}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k + \nu_k \gamma_k' - \nu_k \frac{\sum_{j \in \mathcal{N}} \gamma_j}{H} h_k'\right)\right),$$

$$= \frac{-h_k'}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k + \nu_k\gamma_k' - (\nu_k - 1)h_k'\right)\right),$$
$$= \frac{-h_k'}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k - \frac{\partial \nu_k}{\partial p_k}\gamma_k\right)\right),$$
$$= \frac{h_k'}{H}\frac{\partial \nu_k}{\partial p_k}.$$

where the first line follows from differentiating equation (vii) with respect to p_k and using the fact that $\partial\Pi(M)/\partial p_k = 0$, the second line follows from equation (ix), and the third line follows from equation (vi). Using the same method, we find that all the off-diagonal elements of the Hessian matrix are equal to zero, which proves the lemma. □

The following lemma is an immediate consequence of Lemma III and equation (v):

Lemma IV. *The set $\mathcal{H}^\iota$ is well-behaved.*

Proof. Let $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right) \in \mathcal{M}(\mathcal{H})$. Let $p >> 0$ and $H^0 > 0$, and suppose that $\nabla_p\Pi(M)(p, H^0) = 0$. Then, by equation (ix), and by log-convexity of h_j for every j, $\nu_k(p_k, c_k) > 1$ for every $1 \leq k \leq n$. It follows that $\iota_k(p_k) > 1$ and $p_k > c_k$ for every k. Therefore, by equation (v) and since $h_k \in \mathcal{H}^\iota$, $\partial\nu_k/\partial p_k > 0$. By Lemma III, the Hessian of $\Pi(M)(., H^0)$ evaluated at price vector p is therefore negative definite. Therefore, the local second-order conditions hold, p is a local maximizer of $\Pi(M)(., H^0)$, M is well-behaved, and $\mathcal{H}^\iota$ is well-behaved. □

The next step is to rule out products that are not in $\mathcal{H}^\iota$. This is done in the following lemma:

Lemma V. *Let $h \in \mathcal{H}\backslash\mathcal{H}^\iota$. Then, $\mathcal{H}^{CES} \cup \{h\}$ is not well-behaved.*

Proof. Since $h \notin \mathcal{H}^\iota$, there exists $\hat{p} > 0$ such that $\iota(\hat{p}) > 1$ and $\iota'(\hat{p}) < 0$. Our goal is to construct a two-product firm $M = ((h_1, h_2), (c_1, c_2))$, a price vector $(p_1, p_2) \in \mathbb{R}^2_{++}$ and an $H^0 > 0$ such that $\nabla_p\Pi(M)((p_1, p_2), H^0) = 0$ and $\frac{\partial\nu_1}{\partial p_1}(p_1, c_1) < 0$. We begin by setting $h_1 = h$ and $p_1 = \hat{p}$. We will tweak h_2, p_2, c_1, c_2 and H^0 along the way.

Since $\iota_1'(p_1) < 0$, equation (v) implies that there exists $\bar{c} \in (0, p_1)$ such that $\frac{\partial\nu_1}{\partial p_1}(p_1, c_1) < 0$ whenever $c_1 < \bar{c}$.

For every $s \in (1, \iota_1(p_1))$, there exists a unique $C_1(s) \in (0, p_1)$ such that

$$\frac{p_1 - C_1(s)}{p_1}\frac{\iota_1(p_1)}{s} = 1. \tag{x}$$

$C_1(\cdot)$ is continuous and $\lim_{s\to\iota_1(p_1)} C_1(s) = 0$. In particular, there exists $\underline{s} \in (1, \iota_1(p_1))$ such that $C_1(s) \in (0, \bar{c})$ whenever $s \in (\underline{s}, \iota_1(p_1))$. It follows that, when $s \in (\underline{s}, \iota_1(p_1))$, condition (x) holds and $\frac{\partial\nu_1}{\partial p_1}(p_1, C_1(s)) < 0$.

Let $\sigma \in (\underline{s}, \iota_1(p_1))$, and $h_2(p_2) = p_2^{1-\sigma}$ for all $p_2 > 0$. Recall that $\iota_2(p_2) = \sigma$ and $\gamma_2(p_2) = \frac{\sigma-1}{\sigma}h_2(p_2)$ for all $p_2 > 0$.

For every $H^0 > 0$, define the following function:

$$\phi(x) = 1 - \frac{\gamma_1(p_1) + \frac{\sigma-1}{\sigma}x}{h_1(p_1) + x + H^0}, \ \forall x > 0.$$

Notice that $\lim_{x\to\infty}\phi(x) = \frac{1}{\sigma}$. Moreover,

$$\phi'(x) = \frac{\gamma_1(p_1) - \frac{\sigma-1}{\sigma}\left(h_1(p_1) + H^0\right)}{\left(h_1(p_1) + x + H^0\right)^2}.$$

Choose some H^0 such that $\gamma_1(p_1) - \frac{\sigma-1}{\sigma}\left(h_1(p_1) + H^0\right) < 0$. Then, $\phi'(x) < 0$ for all $x > 0$. Therefore, $\phi(x) > \frac{1}{\sigma}$ for all $x > 0$.

Let $(p_2, c_2) \in \mathbb{R}^2_{++}$. The first-order condition for product 2 can be written as follows:

$$\frac{p_2 - c_2}{p_2}\sigma\left(1 - \frac{\gamma_1(p_1) + \gamma_2(p_2)}{h_1(p_1) + h_2(p_2) + H^0}\right) = 1,$$

or, equivalently,

$$\frac{p_2 - c_2}{p_2} \times \underbrace{\sigma\phi\left(p_2^{1-\sigma}\right)}_{>1,\text{ since }\phi(x)>1/\sigma} = 1.$$

Therefore, for every $p_2 > 0$, there exists a unique $C_2(p_2) \in (0, p_2)$ such that the first-order condition for product 2 holds.

The first-order condition for product 1 can be written as follows:

$$\frac{p_1 - c_1}{p_1}\frac{\iota_1(p_1)}{\phi\left(p_2^{1-\sigma}\right)^{-1}} = 1.$$

Since $\phi\left(p_2^{1-\sigma}\right)^{-1} \underset{p_2\to 0^+}{\longrightarrow} \sigma$ and $\sigma \in (\underline{s}, \iota_1(p_1))$, there exists $P_2 > 0$ such that $\phi\left(P_2^{1-\sigma}\right)^{-1} \in (\underline{s}, \iota_1(p_1))$. Put $c_1 = C_1\left(\phi\left(P_2^{1-\sigma}\right)^{-1}\right)$. Then, the first-order condition for product 1 holds, $c_1 \in (0, \bar{c})$, and therefore, $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$.

To summarize, we have constructed a multi-product firm $M = ((h_1, h_2), (c_1, c_2))$ with $h_1 = h$, $h_2(x) = x^{1-\sigma}$, $c_1 = C_1\left(\phi\left(P_2^{1-\sigma}\right)^{-1}\right)$ and $c_2 = C_2(P_2)$, an $H^0 > 0$ and a price vector $(p_1, p_2) = (\hat{p}, P_2)$ such that $\nabla_p \Pi(M)\left((p_1, p_2), H^0\right) = 0$ and $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$. By Lemma III, the Hessian matrix of $\Pi(M)(\cdot, H^0)$ evaluated at price vector (p_1, p_2) has a strictly positive eigenvalue. Therefore, (p_1, p_2) is not a local maximizer of $\Pi(M)(\cdot, H^0)$, and multi-product firm M is not well-behaved. It follows that $\mathcal{H}^{CES} \cup \{h\}$ is not well-behaved. $\square$

Combining Lemmas IV and V proves Theorem I.

III.3 A Remark on Single-Product Firms

We now argue that multiproduct-firms are special, in the sense that, compared to single-product firms, they require strictly stronger restrictions on the set of admissible products to be well-behaved. This statement is formalized in the following proposition:

Proposition II. *Let $h \in \mathcal{H}$, $c > 0$ and $M = (h, c)$. The following assertions are equivalent:*

(i) Firm M is well-behaved.

(ii) For every $p > 0$ such that $\iota(p) > 1$, $\iota'(p) \geq 0$ or $\rho'(p) \geq 0$.[6]

Proof. Let $h \in \mathcal{H}$, $c > 0$ and $M = (h, c)$. With single-product firms, first-order condition (ix) can be simplified as follows:

$$\nu \left(1 - \frac{\gamma}{h + H^0}\right) = 1. \tag{xi}$$

By Lemma III, $\partial^2 \Pi(M)/\partial p^2$ has the same sign as $\partial \nu / \partial p$ whenever condition (xi) holds.

Assume that (ii) holds. We want to show that, for every $(p, c, H^0) \in \mathbb{R}^3_{++}$, $\partial \nu(p, c)/\partial p > 0$ whenever condition (xi) holds. Let $p > 0$. If $\iota(p) \leq 1$, then for every $c, H^0 > 0$,

$$\nu \left(1 - \frac{\gamma}{h + H^0}\right) < 1,$$

so there is nothing to prove. Next, assume that $\iota(p) > 1$. For every $c > 0$, $\partial \nu / \partial p$ is given by equation (v). If $\iota'(p) \geq 0$, then $\partial \nu(p, c)/\partial p > 0$ for every $H^0 > 0$ and $0 < c \leq p$. In particular, $\partial \nu(p, c)/\partial p > 0$ when condition (xi) holds. (Recall that, by log-convexity, $\gamma < h + H^0$.)

Assume instead that $\iota'(p) < 0$. Then, since (ii) holds, $\rho'(p) \geq 0$. Notice that

$$\frac{\rho'}{\rho} = \left(\log\left(\frac{h\iota}{p(-h')}\right)\right)' = \frac{h'}{h} + \frac{\iota'}{\iota} - \frac{1}{p} + \frac{h''}{-h'}.$$

It follows that

$$p\frac{\rho'}{\rho} = p\frac{\iota'}{\iota} - p\frac{-h'}{h} - 1 + \iota = p\frac{\iota'}{\iota} - \frac{\iota}{\rho} - 1 + \iota = p\frac{\iota'}{\iota} + \iota\left(1 - \frac{1}{\rho}\right) - 1.$$

Since $\iota' < 0$ and $\rho' \geq 0$, it follows that $\iota\left(1 - \frac{1}{\rho}\right) - 1 > 0$.

Since $\iota(p) > 1$, we have that, for every $H^0 > 0$, there exists a unique $c\left(H^0\right)$ such that condition (xi) holds. This $c\left(H^0\right)$ is given by:

$$c\left(H^0\right) = p\left(1 - \frac{1}{\iota\left(1 - \frac{\gamma}{h + H^0}\right)}\right). \tag{xii}$$

[6] Recall that $\rho = h/\gamma$.

Since $\iota\left(1-\frac{1}{\rho}\right)-1>0$, $c\left(H^0\right)\in(0,p)$ for every $H^0>0$. Notice also that $c'\left(H^0\right)>0$. All we need to do now is check that

$$\frac{\partial\nu}{\partial p}\left(p,c\left(H^0\right)\right)=\frac{c\left(H^0\right)}{p^2}\iota+\frac{p-c\left(H^0\right)}{p}\iota'$$

is strictly positive for every $H^0>0$. Since the right-hand side is strictly increasing in $c\left(H^0\right)$ and $c'\left(H^0\right)>0$, this boils down to checking that $\partial\nu\left(p,c(0)\right)/\partial p\geq 0$:

$$\begin{aligned}\frac{\partial\nu}{\partial p}\left(p,c(0)\right)&=\frac{\iota}{p}\left(\frac{c(0)}{p}\iota+\frac{p-c(0)}{p}p\frac{\iota'}{\iota}\right),\\&=\frac{\iota}{p}\left(\left(1-\frac{1}{\iota\left(1-\frac{1}{\rho}\right)}\right)+\frac{1}{\iota\left(1-\frac{1}{\rho}\right)}p\frac{\iota'}{\iota}\right),\\&=\frac{1}{p\left(1-\frac{1}{\rho}\right)}\left(\iota\left(1-\frac{1}{\rho}\right)-1+p\frac{\iota'}{\iota}\right),\\&=\frac{\rho'}{\rho-1},\end{aligned}$$

which is indeed non-negative. Therefore, (i) holds.

Conversely, suppose that (ii) does not hold. There exists $p>0$ such that $\iota(p)>1$, $\iota'(p)<0$ and $\rho'(p)<0$. We distinguish two cases. Assume first that $\iota\left(1-\frac{1}{\rho}\right)-1\geq 0$. Then, the $c\left(H^0\right)$ defined in equation (xii) satisfies $c(H^0)\in(0,p)$ and

$$\frac{p-c\left(H^0\right)}{p}\iota\left(1-\frac{\gamma}{h+H^0}\right)=1$$

for every $H^0>0$. In addition, as proven above,

$$\frac{\partial\nu}{\partial p}\left(p,c(0)\right)=\frac{\rho'}{\rho-1}<0.$$

By continuity, there exists $\varepsilon>0$ such that $\frac{\partial\nu}{\partial p}\left(p,c(\varepsilon)\right)<0$. It follows that $\frac{\partial\Pi(M)}{\partial p}(p,\varepsilon)=0$ and $\frac{\partial^2\Pi(M)}{\partial p^2}(p,\varepsilon)>0$. Therefore, M is not well-behaved.

Next, assume that $\iota\left(1-\frac{1}{\rho}\right)-1<0$. Then, there exists $H^0>0$ such that $c\left(H^0\right)=0$. Notice that $\frac{\partial\nu}{\partial p}\left(p,0\right)=\iota'(p)<0$. Therefore, by continuity of $\partial\nu/\partial p$ and $c(.)$, for $\varepsilon>0$ small enough,

$$\frac{\partial\nu}{\partial p}\left(p,c\left(H^0+\varepsilon\right)\right)<0,$$

and $c\left(H^0+\varepsilon\right)>0$. Therefore, multiproduct firm $\left(h,c\left(H^0+\varepsilon\right)\right)$ is not well-behaved. □

III.4 Equilibrium Existence without Assumption 1

Assumption 1 can be relaxed if we follow instead a potential games approach (Slade, 1994; Monderer and Shapley, 1996). In Nocke and Schutz (2016), we show that the function

$$P(p) = \frac{\prod_{f\in\mathcal{F}} \sum_{j\in f}(p_j - c_j)(-h_j'(p_j))}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}$$

is an ordinal potential for our pricing game. The idea is that, starting from a profile of prices, if firm f deviates, then firm f's profit increases if and only if the value of the potential function increases. Without putting any restrictions on the demand system $((h_j)_{j\in\mathcal{N}}, H^0)$ (except that the h functions are positive, $\mathcal{C}^1$, strictly decreasing and log-convex), we show that the function P has a global maximizer. This implies that the pricing game has an equilibrium.

While this more general existence result is useful, the downside of the potential games approach is that it does not allow us to completely characterize the set of equilibria. This implies in particular that we cannot extend the comparative statics and characterization results derived in Section 3.3

IV Choke Price

In this section, we show how to extend the analysis to the case where (some of the) products have a choke price.

Demand. The demand for product i is still given by $D_i(p) = -h_i'(p_i)/H(p)$, but we now assume that $h_i'(p_i) = 0$ whenever p_i exceeds some choke price $\overline{p}_i \in (0,\infty]$. (Note that, if $\overline{p}_i = \infty$ for every product i, then we have the baseline model studied in the paper.) More precisely, assume that, for every i, there exists $\overline{p}_i \in (0,\infty]$ such that h_i is strictly positive, log-convex, and $\mathcal{C}^1$ on $\mathbb{R}_{++}$, constant on $(\overline{p}_i,\infty)$, and $\mathcal{C}^3$ and strictly decreasing on $(0,\overline{p}_i)$. These assumptions imply that h_i continues to be the exponential of an indirect subutility function. Hence, the demand system $((h_j)_{j\in\mathcal{N}}, H^0)$ can still be given discrete/continuous choice foundations. Moreover, consumer surplus is still given by $\log H(p)$.

The following function h_i satisfies the assumptions made above:

$$h_i(p_i) = \begin{cases} \exp\left(a_i p_i - \frac{1}{2} b_i p_i^2\right) & \text{if } p_i \leq \overline{p}_i = \frac{a_i}{b_i}, \\ \exp\left(\frac{a_i^2}{2b_i}\right) & \text{otherwise.} \end{cases} \tag{xiii}$$

Note that the conditional demand for product i is linear up to the choke price: $-h_i'(p_i)/h_i(p_i) = a_i - b_i p_i$.

The pricing game. A pricing game is still a tuple $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$. The profit of firm $f \in \mathcal{F}$ is now defined as follows:

$$\Pi^f(p) = \sum_{\substack{j\in f\\ p_j<\overline{p}_j}} (p_j - c_j)\frac{-h_j'(p_j)}{\sum_{k\in\mathcal{N}} h_k(p_k) + H^0}, \quad \forall p \in (0,\infty]^{\mathcal{N}}.$$

Let p be a price vector such that $p_j \geq \overline{p}_j$ for every j in some subset of products $\mathcal{N}'$. Note that setting the prices of all the products in $\mathcal{N}'$ equal to the corresponding choke prices while leaving the prices of the other products unchanged affects neither the firms' profits nor consumer surplus. We can therefore restrict the strategy space to $\prod_{j\in\mathcal{N}}(0,\overline{p}_j]$.

For every $p_i \in (0,\overline{p}_i)$, let $\iota_i(p_i) = p_i h_i''(p_i)/(-h_i'(p_i))$ be the price elasticity of demand for product i under monopolistic competition. The following assumption plays the same role as in the paper:

Assumption i. *For every $p_i \in (0,\overline{p}_i)$, $\iota_i'(p_i) \geq 0$ whenever $\iota_i(p_i) > 1$.*

It is easily checked that the function h_i defined in equation (xiii) satisfies this assumption, as long as a_i and b_i are not too different.

Equilibrium analysis. The equilibrium characterization and the proof of equilibrium existence follow the analysis in Sections 3.1, 3.2, and Appendix A very closely.

Note first that, since products are substitutes, pricing below cost is always strictly suboptimal. Hence, if product i is such that $\overline{p}_i \leq c_i$, then firm i optimally sets $p_i = \overline{p}_i$. We can therefore remove product i from the set of products, redefine H^0 as $H^0 + h_i(\overline{p}_i)$, and obtain a pricing game that is formally equivalent to the original one. Having done that for every product for which the production cost exceeds the choke price, we obtain a new set of products $\mathcal{N}$, a new set of firms $\mathcal{F}$, and a new value for the outside option H^0, such that $\overline{p}_j > c_j$ for every $j \in \mathcal{N}$. We study this modified pricing game in the following.

It is straightforward to show that each firm sets at least one price below the choke price in any equilibrium (Lemma B). Since pricing below cost is strictly suboptimal, we can restrict the strategy space to $\prod_{j\in\mathcal{N}}[c_j,\overline{p}_j]$. The continuity and compactness argument used in the proof of Lemma C therefore still goes through, implying that, holding the prices of firm f's rivals fixed, firm f's profit maximization problem has a solution.

The definition of generalized first-order conditions has to be modified to account for the fact that some of the choke prices may be finite. As in the paper, let $G^f((p_j)_{j\in f}, H^{0\prime})$ be the profit of firm f when it chooses the profile of prices $(p_j)_{j\in f}$ and its rivals' contribution to the aggregator is $H^{0\prime}$. $\left(p_k, (p_j)_{j\in f\setminus\{k\}}\right)$ denotes the price vector with k-th component p_k, and with other components given by $(p_j)_{j\in f\setminus\{k\}}$. We say that the generalized first-order conditions of the maximization problem $\max G^f(\cdot, H^{0\prime})$ hold at price vector $(\tilde{p}_j)_{j\in f} \in \prod_{j\in f}[c_j,\overline{p}_j]$ if for every $k \in f$,

(a) $\frac{\partial G^f}{\partial p_k}((\tilde{p}_j)_{j\in f}, H^{0\prime}) = 0$ whenever $\tilde{p}_k < \overline{p}_k$, and

(b) $G^f\left((\tilde{p}_j)_{j\in f}, H^{0\prime}\right) \geq G^f\left(\left(p_k, (\tilde{p}_j)_{j\in f\setminus\{k\}}\right), H^{0\prime}\right)$ for every $p_k < \overline{p}_k$ whenever $\tilde{p}_k = \overline{p}_k$.

Generalized first-order conditions are clearly necessary for optimality (Lemma D).

We now extend the definition of the pricing function r_j to the case of finite choke prices (Lemma E). Let $\nu_j(p_j) = \frac{p_j - c_j}{p_j}\iota_j(p_j)$. The argument in the proof of Lemma A can be easily extended to show that, for every j, there exists $\underline{p}_j \in (0, \overline{p}_j)$ such that $\iota_j(p_j) > 1$ if and only if $p_j \in (\underline{p}_j, \overline{p}_j)$. Next, we show that p_j^{mc}, the price of product j under monopolistic competition, which solves the equation $\nu_j(p_j) = 1$ on interval $(0, \overline{p}_j)$, is well defined when the choke price is finite. Assume first that the equation has no solution. Since $\nu_j(p_j) < 1$ for p_j sufficiently close to c_j, the continuity of ι_j implies that $\nu_j(p_j) < 1$ for every $p_j \in (0, \overline{p}_j)$. It follows that $(p_j - c_j)(-h_j'(p_j))$ is strictly increasing on $(0, \overline{p}_j)$. The fact that $(\overline{p}_j - c_j)(-h_j'(\overline{p}_j)) = 0$ gives us a contradiction. Next, note that, by definition of $\underline{p}_j$, any solution to the equation $\nu_j(p_j) = 1$ has to belong to the interval $(\underline{p}_j, \overline{p}_j)$. Since $\nu_j(\cdot)$ is strictly increasing on that interval, it follows that the solution is unique.

We can now extend Lemma E: ν_j is a strictly increasing $\mathcal{C}^1$-diffeomorphism from $(p_j^{mc}, \overline{p}_j)$ to $(1, \bar{\mu}_j)$, where $\bar{\mu}_j \equiv \lim_{p_j \to \overline{p}_j^-} \nu_j(p_j) > 1$. The corresponding inverse function, r_j, is therefore strictly increasing from $(1, \bar{\mu}_j)$ to $(p_j^{mc}, \overline{p}_j)$. The derivative of r_j is still given by equation (11). As in the paper, we extend the functions ν_k and r_k by continuity as follows: $\nu_k(\overline{p}_k) = \bar{\mu}_k$, $r_k(1) = p_k^{mc}$, and $r_k(\mu^f) = \overline{p}_k$ for every $\mu^f \geq \bar{\mu}_k$. We also extend γ_k by continuity at $\overline{p}_k$: $\gamma_k(\infty) = 0$.[7]

Having extended the definition of pricing functions to accommodate finite choke prices, we can define the common ι-markup property. A profile of prices $(p_j)_{j\in f} \in \prod_{j\in f}[c_j, \overline{p}_j]$ satisfies that property if there exists $\mu^f \in (1, \bar{\mu}^f)$ (where $\bar{\mu}^f = \max_{j\in f} \bar{\mu}_j$) such that $p_j = r_j(\mu^f)$ for every $j \in f$. The argument in the proof of Lemma F continues to apply, implying that, if a profile of prices $(p_j)_{j\in f}$ solves firm f's profit maximization problem, then it must satisfy the common ι-markup property, and the corresponding ι-markup must solve equation (12). The argument used in the proof of Lemma G (recall that $\gamma_j(\overline{p}_j) = 0$ for every j) implies that that equation has a unique solution. This allows us to generalize Lemma H, and to conclude our study of firm f's profit maximization problem: The generalized first-order conditions are necessary and sufficient for global optimality, and the optimal ι-markup is the unique solution of equation (12).

Having shown that first-order conditions are sufficient for global optimality, we can use an aggregative games approach to prove equilibrium existence and characterize the set of equilibria. The monotonicity of γ_j and r_j and the fact that $\gamma_j(\overline{p}_j) = 0$ for every j imply that equation (14) has a unique solution (Lemma I). Therefore, the fitting-in function $m^f(H)$ is well defined, continuous, strictly decreasing, and satisfies $\lim_{H\to 0} m^f(H) = \bar{\mu}^f$ and

[7] We already know form Lemma A that $\lim_{p_k \to \overline{p}_k} \gamma_k(p_k) = 0$ if $\overline{p}_k = \infty$. Suppose $\overline{p}_k < \infty$. Then,

$$\lim_{p_k \to \overline{p}_k} \gamma_k(p_k) = \lim_{p_k \to \overline{p}_k} p_k \frac{-h_k'(p_k)}{\iota_k(p_k)} = \underbrace{\lim_{p_k \to \overline{p}_k} p_k}_{<\infty} \times \underbrace{\lim_{p_k \to \overline{p}_k} -h_k'(p_k)}_{=0} \times \underbrace{\lim_{p_k \to \overline{p}_k} \frac{1}{\iota_k(p_k)}}_{<\infty} = 0.$$

$\lim_{H\to\infty} m^f(H) = 1$. The equilibrium existence and characterization problem therefore boils down to identifying the set of H's such that $\Omega(H) = 1$, where

$$\Omega(H) \equiv \frac{H^0}{H} + \frac{1}{H}\sum_{f\in\mathcal{F}}\sum_{j\in f} h_j(r_j(m^f(H)))$$

is the aggregate share function.

If all the products have infinite choke prices, then we already know from Lemma J that equation $\Omega(H) = 1$ has a solution. Suppose that $\overline{p}_j < \infty$. Then,

$$\Omega(H) \geq \frac{H^0 + h_j(\overline{p}_j)}{H} \underset{H\to 0}{\longrightarrow} \infty.$$

The fact that $\Omega(H) \underset{H\to\infty}{\longrightarrow} 0$ (as shown in the proof of Lemma J) and the continuity of Ω allow us to conclude that equation $\Omega(H) = 1$ has a solution.

Therefore, Theorem 1 extends to the case of finite choke prices. The set of equilibrium aggregator levels is still the set of fixed points of the aggregate fitting-in function. For a given equilibrium aggregator level H^*, firm f sets a ι-markup of $\mu^{f*} = m^f(H^*)$, and earns a profit of $\mu^{f*} - 1$. Product $j \in f$ is priced at $r_j(\mu^{f*})$. The fact that fitting-in functions and pricing functions have the same monotonicity properties as in the paper implies that the comparative statics results derived in Section 3.3 continue to hold. In particular, a shock that makes the industry more competitive (say, higher H^0) induces firms to lower their prices and broaden their scope in the highest and lowest equilibrium.

V Equilibrium Uniqueness

V.1 Main Results

Fix a pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ satisfying Assumption 1. We now study equilibrium uniqueness by deriving conditions under which the function $\Omega(H) = \Gamma(H)/H$ is strictly decreasing in H.[8] We recall the following notation: For all $j \in \mathcal{N}$, $\gamma_j = h_j'^2/h_j''$, $\rho_j \equiv h_j/\gamma_j$, and $\underline{p}_j = \inf\{p_j > 0 : \ \iota_j(p_j) > 1\}$. For every $j \in \mathcal{N}$ and $p_j > \underline{p}_j$, let $\theta_j(p_j) = h_j'(p_j)/\gamma_j'(p_j)$.

We can now state our uniqueness theorem:

Theorem II. *Let $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying Assumption 1. Suppose that, for every firm $f \in \mathcal{F}$, at least one of the following conditions holds:*

(a) $\min_{j\in f} \inf_{p_j>\underline{p}_j} \rho_j(p_j) \geq \max_{j\in f} \sup_{p_j>\underline{p}_j} \theta_j(p_j)$.

[8] Another possibility would be to follow an index approach and compute the sign of the determinant of the Jacobian of the first-order conditions map. In Section V.5, we show that this approach delivers the same uniqueness conditions.

(b) *$\bar{\mu}^f \leq \mu^*(\simeq 2.78)$, and for every $j \in f$, $\bar{\mu}_j = \bar{\mu}^f$, $\lim_\infty h_j = 0$ and ρ_j is non-decreasing on $(\underline{p}_j, \infty)$.*[9]

(c) *There exist a function h^f, a marginal cost level $c^f > 0$, and a collection of quality shifters $(a_j)_{j\in f} \in \mathbb{R}^f_{++}$ such that $h_j = a_j h^f$ and $c_j = c^f$ for all $j \in f$. In addition, ρ^f is non-decreasing on $(\underline{p}, \infty)$.*

Then, the pricing game has a unique equilibrium.

Proof. See Section V.2. □

As already mentioned in the paper, the condition that ρ_j is non-decreasing is equivalent to the reciprocal of the demand function $p_j \in (\underline{p}_j, \infty) \mapsto \widehat{D}_j(p_j, h_j(p_j) + H^0)$ being convex for every $H^0 > 0$.[10] This convexity condition guarantees equilibrium uniqueness, provided that some additional restrictions, contained in conditions (a), (b) and (c), are satisfied. Note that condition (a) is indeed a stronger version of the assumption that ρ_j is non-decreasing. This is because ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ if and only if $\rho_j \geq \theta_j$ on the same interval.[11] Condition (a) imposes that the highest possible value of θ_j $(j \in f)$ be smaller than the lowest possible value of ρ_j $(j \in f)$, which is indeed stronger.

In Section VI.2, we provide examples of functional forms that satisfy (or do not satisfy) our uniqueness conditions. There, we also develop a cookbook for applied work.

Some pricing games satisfy none of our uniqueness conditions. In such cases, it is still possible to establish equilibrium uniqueness, provided that the firms are sufficiently inefficient and/or consumers have access to a sufficiently attractive outside option:

Proposition III. *Suppose that $(h_j)_{j\in\mathcal{N}}$ satisfies Assumption 1, and let $\mathcal{F}$ be a firm partition. Then,*

- *For every $\underline{H}^0 > 0$, there exists $\underline{c} > 0$ such that the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium whenever $(c_j)_{j\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$ and $H^0 \geq \underline{H}^0$.*
- *For every $\underline{c} > 0$, there exists $\underline{H}^0 \geq 0$ such that the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium whenever $(c_j)_{j\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$ and $H^0 \geq \underline{H}^0$.*

Proof. See Section V.4. □

[9] Condition $\lim_{p_j\to\infty} h_j(p_j) = 0$ can be weakened. See Propositions IV and V, and Corollaries II and III in Section V.3.

[10] To see this, note that

$$\frac{d^2}{dp_j^2}\frac{1}{\widehat{D}_j} = -\left(\frac{h_j + H^0}{h'_j}\right)'' = -\left(\frac{h_j'^2 - h''_j(h_j + H^0)}{h_j'^2}\right)' = \left(\rho_j + \frac{H^0}{\gamma_j}\right)' = \rho'_j - H^0\frac{\gamma'_j}{\gamma_j}.$$

Since $\gamma'_j < 0$ (see Lemma A), the above expression is non-negative for every H^0 if and only if $\rho'_j \geq 0$.

[11] To see this, note that $(\log \rho_j)' = \frac{-\gamma'_j}{h_j}(\rho_j - \theta_j)$, and that $\gamma'_j < 0$ by Lemma A.

Intuitively, when the products in $\mathcal{N}$ are relatively unattractive compared to the outside option (either because marginal costs are high, or because the outside option delivers high consumer surplus), the firms have low market shares, and, hence, little market power. The firms therefore set ι-markups close to those they would set under monopolistic competition, and react relatively little to changes in their rivals' behavior.

V.2 Proof of Theorem II

V.2.1 Preliminaries

The following lemma allows us to study the equilibrium uniqueness problem on a firm-by-firm basis:

Lemma VI. *Let $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying Assumption 1. Suppose that, for every $f \in \mathcal{F}$, the function*

$$s^f: \; \mu^f \in (1, \bar{\mu}^f) \mapsto \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j\in f} h_j(r_j(\mu^f))}{\sum_{j\in f} \gamma_j(r_j(\mu^f))}$$

is strictly increasing in μ^f. Then, the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium.

Proof. A sufficient condition for the pricing game to have a unique equilibrium is that the function Ω is strictly decreasing. Recall that

$$\begin{aligned}
\Omega(H) &= \frac{H^0}{H} + \sum_{f\in\mathcal{F}} \frac{\sum_{j\in f} h_j\left(r_j(m^f(H))\right)}{H}, \\
&= \frac{H^0}{H} + \sum_{f\in\mathcal{F}} \frac{m^f(H) - 1}{m^f(H)} \frac{\sum_{j\in f} h_j\left(r_j(m^f(H))\right)}{\sum_{j\in f} \gamma_j\left(r_j(m^f(H))\right)}, \\
&= \frac{H^0}{H} + \sum_{f\in\mathcal{F}} s^f\left(m^f(H)\right),
\end{aligned}$$

where the second line follows by equation (14) in the paper. Combining this with the fact that m^f is strictly decreasing for every f (see Lemma I in the paper) proves the lemma. □

All we need to do now is show that, if condition (a), (b) or (c) in Theorem II holds for firm f, then s^f is strictly increasing. We do so in Sections V.2.2 and V.2.3.

V.2.2 Sufficiency of Conditions (a) and (c)

We first show that condition (a) is sufficient for s^f to be strictly increasing.

Lemma VII. *Suppose condition (a) in Theorem II holds for firm $f \in \mathcal{F}$. Then, the function s^f defined in Lemma VI is strictly increasing. Moreover, $s^{f\prime}(\mu^f) > 0$ for every $\mu^f \in (1, \bar{\mu}^f) \setminus \{\bar{\mu}_j\}_{j \in f}$.*

Proof. By Lemma E in the paper, s^f is continuous on $(1, \bar{\mu}^f)$ and $\mathcal{C}^1$ on $(1, \bar{\mu}^f) \setminus \{\bar{\mu}_j\}_{j \in f}$. To show that s^f is strictly increasing, it is therefore enough to show that $s^{f\prime}(\mu^f) > 0$ for every $\mu^f \notin \{\bar{\mu}_j\}_{j \in f}$. Fix such a μ^f. Let f' be the set of $j's$ such that $\mu^f > \bar{\mu}_j$. Then, since $\gamma_j(\infty) = 0$ for every j (see Lemma A),

$$s^f(\mu^f) = \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f'} \lim_{p_j \to \infty} h_j(p_j)}{\sum_{j \notin f'} \gamma_j(r_j(\mu^f))} + \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \notin f'} h_j(r_j(\mu^f))}{\sum_{j \notin f'} \gamma_j(r_j(\mu^f))}.$$

Since γ_j is strictly decreasing and r_j is strictly increasing for every j (see Lemmas A and E), the first term in the above expression is non-decreasing. We now turn our attention to the second term. Note that

$$\begin{aligned}
\left(\frac{\sum_{j \notin f'} h_j(r_j(\mu^f))}{\sum_{j \notin f'} \gamma_j(r_j(\mu^f))} \right)' &= \frac{\sum_{j,k \notin f'} r_j'(h_j' \gamma_k - \gamma_j' h_k)}{\left(\sum_{j \notin f'} \gamma_j \right)^2}, \\
&= \frac{\sum_{j,k \notin f'} \gamma_k (-\gamma_j') r_j' (\rho_k - \theta_j)}{\left(\sum_{j \notin f'} \gamma_j \right)^2},
\end{aligned}$$

which is non-negative, since condition (a) holds. (Note that, for every j, $r_j(\mu^f) > p_j^{mc} > \underline{p}_j$.) Since $(\mu^f - 1)/\mu^f$ has a strictly positive derivative, it follows that $s^{f\prime}(\mu^f) > 0$. □

Next, we investigate the sufficiency of condition (c):

Lemma VIII. *Suppose condition (c) in Theorem II holds for firm $f \in \mathcal{F}$. Then, the function s^f defined in Lemma VI is strictly increasing. Moreover, $s^{f\prime}(\mu^f) > 0$ for every $\mu^f \in (1, \bar{\mu}^f)$.*

Proof. It is straightforward to check that, for every $j \in f$, $\iota_j = \iota^f$ and $\gamma_j = a_j \gamma^f$. The fact that $\iota_j = \iota^f$ and $c_j = c^f$ for every j immediately implies that $\bar{\mu}_j = \bar{\mu}^f$ and $r_j = r^f$ for every j. Hence, s^f can be simplified as follows:

$$s^f(\mu^f) = \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f} a_j h^f(r^f(\mu^f))}{\sum_{j \in f} a_j \gamma^f(r^f(\mu^f))} = \frac{\mu^f - 1}{\mu^f} \rho^f(r^f(\mu^f)).$$

Hence, $s^{f\prime}(\mu^f) > 0$. □

V.2.3 Sufficiency of Condition (b)

The goal of this section is to prove the following lemma:

Lemma IX. *Suppose condition (b) in Theorem II holds for firm $f \in \mathcal{F}$. Then, the function s^f defined in Lemma VI is strictly increasing. Moreover, $s^{f\prime}(\mu^f) > 0$ for every $\mu^f \in (1, \bar{\mu}^f)$.*

The proof of Lemma IX proceeds in several steps. We first introduce new notation: $\omega^f = (\mu^f - 1)/\mu^f$, $\bar{\omega}^f = \lim_{\mu^f \to \bar{\mu}^f} (\mu^f - 1)/\mu^f$, and, for every $j \in f$ and $p_j > \underline{p}_j$, $\chi_j(p_j) = (\iota_j(p_j) - 1)/\iota_j(p_j)$. The following lemma is useful to understand our uniqueness conditions:

Lemma X. *Suppose Assumption 1 holds for firm f. For every $j \in f$:*

- *For every $p_j > \underline{p}_j$, $1 - \theta_j(p_j)\chi_j(p_j) \geq 0$.*
- *For every $\omega^f \in (0, \bar{\omega}^f)$ and $p_j > \underline{p}_j$ such that $\chi_j(p_j) > \omega^f$, $1 - \omega^f \theta_j(p_j) > 0$.*
- *For every $\omega^f \in (0, \bar{\omega}^f)$ and $p_j \geq r_j(1/(1-\omega^f))$, $1 - \omega^f \theta_j(p_j) > 0$.*

Proof. Fix some j in f. Since $\iota_j(p_j) = p_j(-h_j'(p_j))/\gamma_j(p_j)$, we have that, for every $p_j > \underline{p}_j$,

$$
\begin{aligned}
\frac{\iota_j'(p_j)}{\iota_j(p_j)} &= \frac{1}{p_j}\left(1 - \iota_j(p_j) + p_j \frac{-\gamma_j'(p_j)}{\gamma_j(p_j)}\right), \\
&= \frac{1}{p_j}\left(1 - \iota_j(p_j) + \frac{1}{\theta_j(p_j)} p_j \frac{-h_j'(p_j)}{\gamma_j(p_j)}\right), \\
&= \frac{\iota_j(p_j)}{p_j \theta_j(p_j)} \left(1 - \theta_j(p_j)\chi_j(p_j)\right),
\end{aligned}
$$

which is non-negative by Assumption 1. This proves the first part of the lemma. The second part follows trivially. To prove the third part, note that $p_j \geq r_j\left(\frac{1}{1-\omega^f}\right)$ implies that $\frac{p_j - c_j}{p_j}\iota_j(p_j) \geq \frac{1}{1-\omega^f}$. Hence, $\frac{1}{1-\chi_j(p_j)} = \iota_j(p_j) > \frac{1}{1-\omega^f}$, and $\chi_j(p_j) > \omega^f$. The second part can then be used to obtain the third part. □

We now differentiate the function s^f to obtain conditions under which it is strictly increasing:

Lemma XI. *Suppose that Assumption 1 holds for firm f, and that $\bar{\mu}_j = \bar{\mu}^f$ for every $j \in f$. A sufficient condition for s^f to have a strictly positive derivative on $(1, \bar{\mu}^f)$ is that*

$$
\begin{aligned}
&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall \left(p_j\right)_{j \in f} \in \mathbb{R}^f_{++} \ s.t.\ \forall j \in f,\ \chi_j(p_j) > \omega^f, \\
&\sum_{i,j \in f} \gamma_i(p_i)\gamma_j(p_j)\left(\omega^f \theta_i(p_i) \frac{1 - \omega^f \rho_j(p_j)}{1 - \omega^f \theta_i(p_i)} - \rho_j(p_j)\right) < 0.
\end{aligned} \tag{xiv}
$$

Proof. Since $\bar{\mu}_j = \bar{\mu}^f$ for every $j \in f$, s^f is $\mathcal{C}^1$ on $(1, \bar{\mu}^f)$. For every $\omega^f \in (0, \bar{\omega}^f)$, define $\tilde{s}^f(\omega^f) = s^f(1/(1-\omega^f))$, and, for every $j \in f$, $\tilde{r}_j(\omega^f) = r_j(1/(1-\omega^f))$. Clearly, $s^{f\prime} > 0$ if and only if $\tilde{s}^{f\prime} > 0$. Note that

$$
\tilde{s}^f(\omega^f) = \omega^f \frac{\sum_{j \in f} h_j(\tilde{r}_j(\omega^f))}{\sum_{j \in f} \gamma_j(\tilde{r}_j(\omega^f))}.
$$

Moreover, by Lemma E, we have that

$$\begin{aligned}\tilde{r}_j'(\omega^f) &= \frac{1}{(1-\omega^f)^2} r_j'\left(\frac{1}{1-\omega^f}\right),\\ &= \frac{1}{1-\omega^f}\frac{\gamma_j(\tilde{r}_j(\omega^f))}{-\gamma_j'(\tilde{r}_j(\omega^f)) - \omega^f(-h_j'(\tilde{r}_j(\omega^f)))},\\ &= \frac{1}{1-\omega^f}\frac{\gamma_j(\tilde{r}_j(\omega^f))}{-\gamma_j'(\tilde{r}_j(\omega^f))}\frac{1}{1-\omega^f\theta_j(\tilde{r}_j(\omega^f))}.\end{aligned}$$

We can now compute the elasticity of $\tilde{s}^f$:

$$\begin{aligned}\frac{d\log\tilde{s}^f}{d\log\omega^f} &= 1 + \frac{\omega^f}{1-\omega^f}\sum_{j\in f}\frac{1}{1-\omega^f\theta_j}\frac{\gamma_j}{-\gamma_j'}\left(\frac{-\gamma_j'}{\sum_{k\in f}\gamma_k} - \frac{-h_j'}{\sum_{k\in f}h_k}\right),\\ &= 1 + \frac{\omega^f}{1-\omega^f}\sum_{j\in f}\frac{\gamma_j}{1-\omega^f\theta_j}\left(\frac{1}{\sum_{k\in f}\gamma_k} - \frac{\theta_j}{\sum_{k\in f}h_k}\right).\end{aligned}$$

This elasticity is strictly positive if and only if

$$\begin{aligned}0 &< \sum_{i,j\in f}\left((1-\omega^f)\gamma_i h_j + \omega^f\frac{\gamma_i}{1-\omega^f\theta_i}(h_j - \theta_i\gamma_j)\right),\\ &= \sum_{i,j\in f}\gamma_i\gamma_j\left((1-\omega^f)\rho_j + \frac{\omega^f}{1-\omega^f\theta_i}(\rho_j-\theta_i)\right),\\ &= \sum_{i,j\in f}\gamma_i\gamma_j\left(\rho_j - \omega^f\theta_i\frac{1-\omega^f\rho_j}{1-\omega^f\theta_i}\right),\end{aligned}$$

where, for every $k \in f$, the functions γ_k, ρ_k and θ_k are evaluated at $p_k = \tilde{r}_k(\omega^f)$, which is strictly greater than ω^f (see the argument at the end of the proof of Lemma X). We can therefore use condition (xiv) to conclude that $\tilde{s}^{f\prime}(\omega^f) > 0$. □

The following lemma gives us upper and lower bounds on the function ρ_j ($j \in f$), which will be useful to prove Lemma IX:

Lemma XII. *Suppose that firm f satisfies Assumption 1, and that, for every $j \in f$, ρ_j is non-decreasing on $(\underline{p}_j, \infty)$, $\lim_{p_j\to\infty} h_j(p_j) = 0$, and $\bar{\mu}_j = \bar{\mu}^f < \infty$. Then, for every $\omega^f \in (0, \bar{\omega}^f)$, $k \in f$, and $p_k > 0$ such that $\chi_k(p_k) > \omega^f$,*

$$\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f} \le \rho_k(p_k) \le \frac{1}{\bar{\omega}^f}.$$

Proof. Let $k \in f$ and $\omega^f \in (0, \bar{\omega}^f)$. By Lemma A-(f), $\lim_{p_k\to\infty}\rho_k(p_k) = \frac{\bar{\mu}^f}{\bar{\mu}^f-1} = \frac{1}{\bar{\omega}^f}$. In addition, ρ_k is non-decreasing. Therefore, $\rho_k(p_k) \le \frac{1}{\bar{\omega}^f}$ for all $p_k > \underline{p}_k$. In particular, this inequality is also satisfied if p_k is such that $\chi_k(p_k) > \omega^f$.

In addition, $\rho_k(p_k) = \iota_k(p_k)\frac{h_k(p_k)}{-p_k h'_k(p_k)}$. Therefore,

$$\begin{aligned}\frac{d\log\rho_k(p_k)}{dp_k} &= \frac{\iota'_k(p_k)}{\iota_k(p_k)} + \left(\frac{h'_k(p_k)}{h_k(p_k)} - \frac{1}{p_k} + \frac{h''_k(p_k)}{-h'_k(p_k)}\right),\\ &= \frac{\iota'_k(p_k)}{\iota_k(p_k)} + \frac{1}{p_k}\left(-\frac{\iota_k(p_k)}{\rho_k(p_k)} - 1 + \iota_k(p_k)\right),\\ &= \frac{\iota'_k(p_k)}{\iota_k(p_k)} + \frac{\iota_k(p_k)}{p_k\rho_k(p_k)}\left(\rho_k(p_k)\chi_k(p_k) - 1\right),\\ &\le \frac{\iota'_k(p_k)}{\iota_k(p_k)},\end{aligned}$$

where the last inequality follows from the fact that $\chi_k(p_k) \le \bar{\omega}^f$ and $\rho_k(p_k) \le \frac{1}{\bar{\omega}^f}$. Therefore, for all $p_k > \underline{p}_k$,

$$\log\left(\frac{1}{\bar{\omega}^f\rho_k(p_k)}\right) = \int_{p_k}^{\infty}\frac{\rho'_k(t)}{\rho_k(t)}dt \le \int_{p_k}^{\infty}\frac{\iota'_k(t)}{\iota_k(t)}dt = \log\left(\frac{\bar{\mu}^f}{\iota_k(p_k)}\right) = \log\left(\frac{1-\chi_k(p_k)}{1-\bar{\omega}^f}\right).$$

It follows that,

$$\rho_k(p_k) \ge \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\chi_k(p_k)},\ \forall p_k > \underline{p}_k.$$

In particular, if $\chi_k(p_k) > \omega^f$, then

$$\rho_k(p_k) \ge \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}.$$

□

We now study a maximization problem which will be useful to prove Lemma IX:

Lemma XIII. *For every $\bar{\omega} \in (0,1]$, for every $\omega \in (0,\bar{\omega})$, define*

$$\phi_{\omega,\bar{\omega}} : (y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2 \mapsto \omega y\frac{1-\omega z}{1-\omega y} + \omega z\frac{1-\omega y}{1-\omega z} - y - z.$$

There exists a threshold $\omega^* \in (0,1)$ ($\omega^* \simeq 0.64$) such that if $\bar{\omega} \le \omega^*$, then $\phi_{\omega,\bar{\omega}} \le 0$ for all $\omega \in (0,\bar{\omega})$.

Proof. Let $\bar{\omega} \in (0,1)$ and $\omega \in (0,\bar{\omega})$. Define

$$M(\omega,\bar{\omega}) = \max_{(y,z)\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]^2}\phi_{\omega,\bar{\omega}}(y,z).$$

Notice that $\phi_{\omega,\bar{\omega}}(y,z) = \phi_{\omega,\bar{\omega}}(z,y)$ for every y and z. It follows that

$$M(\omega,\bar{\omega}) = \max_{\substack{(y,z)\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]^2\\ y\le z}}\phi_{\omega,\bar{\omega}}(y,z).$$

Let $\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} \leq y \leq z \leq \frac{1}{\bar{\omega}}$. Then,

$$\begin{aligned}
\frac{\partial \phi_{\omega,\bar{\omega}}}{\partial y} &= \frac{\omega(1-\omega z)}{(1-\omega y)^2} - \frac{\omega^2 z}{1-\omega z} - 1, \\
&= \frac{1}{1-\omega z}\left(\omega\left(\frac{1-\omega z}{1-\omega y}\right)^2 - \omega^2 z - (1-\omega z)\right), \\
&\leq \frac{1}{1-\omega z}\left(\omega - \omega^2 z - (1-\omega z)\right), \text{ since } y \leq z, \\
&= \omega - 1 < 0.
\end{aligned}$$

It follows that, for every $(y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2$ such that $y \leq z$,

$$\phi_\omega(y,z) \leq \phi_\omega\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, z\right) \equiv \psi_{\omega,\bar{\omega}}(z).$$

Therefore,

$$M\left(\omega,\bar{\omega}\right) = \max_{z\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]} \psi_{\omega,\bar{\omega}}(z).$$

Since

$$\psi''_{\omega,\bar{\omega}}(z) = \left(1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right)\frac{2\omega^2}{(1-\omega z)^3} > 0,$$

the function $\psi_{\omega,\bar{\omega}}(.)$ is strictly convex. Therefore,

$$M\left(\omega,\bar{\omega}\right) = \max\left\{\phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}\right), \phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right)\right\}.$$

Since $\phi_{\omega,\bar{\omega}}(z,z) = 2(\omega-1)z < 0$ for every z, it follows that $M\left(\omega,\bar{\omega}\right) \leq 0$ if and only if $\zeta(\omega,\bar{\omega}) \leq 0$, where

$$\begin{aligned}
\zeta(\omega,\bar{\omega}) &\equiv \phi\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right), \\
&= \left(1 - \frac{\omega}{\bar{\omega}}\right)\frac{\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}}{1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}} + \frac{\omega}{\bar{\omega}-\omega}\left(1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right) - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{\omega(1-\bar{\omega})}{\bar{\omega}} + \frac{\omega}{(1-\omega)\bar{\omega}} - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{1}{1-\omega} + \frac{\omega-2}{\bar{\omega}} - \omega.
\end{aligned}$$

For every $\omega \in (0,\bar{\omega})$,

$$\frac{\partial \zeta}{\partial \omega} = \frac{1}{(1-\omega)^2} + \frac{1}{\bar{\omega}} - 1 > 0.$$

Therefore, ζ is strictly increasing in ω on the interval $(0,\bar{\omega})$. It follows that $M\left(\omega,\bar{\omega}\right) \leq 0$ for every $\omega \in (0, \bar{\omega})$ if and only if $\xi(\bar{\omega}) \leq 0$, where

$$\begin{aligned}\xi(\bar{\omega}) &\equiv \zeta(\bar{\omega}, \bar{\omega}), \\ &= \frac{1}{1-\bar{\omega}} + 1 - \bar{\omega} - \frac{2}{\bar{\omega}}.\end{aligned}$$

For every $\bar{\omega} \in (0,1)$,

$$\xi'(\bar{\omega}) = \frac{1}{(1-\bar{\omega})^2} + \frac{2}{(\bar{\omega})^2} - 1 > 0.$$

Therefore, ξ is strictly increasing on $(0,1)$. Since $\lim_{\bar{\omega}\to 0^+} \xi(\bar{\omega}) = -\infty$ and $\lim_{\bar{\omega}\to 1^-} \xi(\bar{\omega}) = +\infty$, there exists a unique threshold $\omega^* \in (0,1)$ such that $\xi(\bar{\omega}) \leq 0$ if and only if $\bar{\omega} \leq \omega^*$. Numerically, we find that $\omega^* \simeq 0.64$. □

We can now prove Lemma IX:

Proof. Suppose condition (b) holds for firm f. Then, $\omega^f < \omega^*$. Splitting the sum in two terms, condition (xiv) in Lemma XI can be rewritten as follows:

$$\begin{aligned}&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (p_j)_{j\in f} \in \mathbb{R}^f_{++} \text{ s.t. } \forall j \in f,\ \chi_j(p_j) > \omega^f, \\ &\frac{1}{2}\sum_{\substack{i,j\in f \\ i\neq j}} \gamma_i(p_i)\gamma_j(p_j)\left(\omega^f\theta_i(p_i)\frac{1-\omega^f\rho_j(p_j)}{1-\omega^f\theta_i(p_i)} + \omega^f\theta_j(p_j)\frac{1-\omega^f\rho_i(p_i)}{1-\omega^f\theta_j(p_j)} - \rho_i(p_i) - \rho_j(p_j)\right) \\ &+ \left(\sum_{i\in f}\gamma_i(p_i)^2\left(\omega^f\theta_i(p_i)\frac{1-\omega^f\rho_i(p_i)}{1-\omega^f\theta_i(p_i)} - \rho_i(p_i)\right)\right) < 0.\end{aligned} \tag{xv}$$

Let us first show that the second sum in equation (xv) is strictly negative. Let $\omega^f \in (0, \bar{\omega}^f)$, $i \in f$ and x_i such that $\chi_i(p_i) > \omega^f$. Then,

$$\omega^f\theta_i(p_i)\frac{1-\omega^f\rho_i(p_i)}{1-\omega^f\theta_i(p_i)} - \rho_i(p_i) \leq \omega^f\theta_i(p_i) - \rho_i(p_i) < 0,$$

where we have used the fact that ρ_i is non-decreasing ($\theta_i(p_i) \leq \rho_i(p_i)$) and Lemma X ($1 - \omega^f\theta_i(p_i) > 0$).

Next, we turn our attention to the first sum in equation (xv). Let $\omega^f \in (0, \bar{\omega}^f)$ and $(p_j)_{j\in f}$ such that $\chi_j(p_j) > \omega^f$ for every $j \in f$. By Lemma XII,

$$\forall k \in f,\ \rho_k(p_k) \in \left[\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}, \frac{1}{\bar{\omega}^f}\right].$$

In addition, as shown above, for every $k \in f$, $\theta_k(p_k) \leq \rho_k(p_k)\ \left(< \frac{1}{\omega^f}\right)$. Therefore,

$$\frac{1}{2}\sum_{\substack{i,j\in f \\ i\neq j}} \gamma_i(p_i)\gamma_j(p_j)\left(\omega^f\theta_i(p_i)\frac{1-\omega^f\rho_j(p_j)}{1-\omega^f\theta_i(p_i)} + \omega^f\theta_j(p_j)\frac{1-\omega^f\rho_i(p_i)}{1-\omega^f\theta_j(p_j)} - \rho_i(p_i) - \rho_j(p_j)\right)$$

$$\leq \frac{1}{2} \sum_{\substack{i,j \in f \\ i \neq j}} \gamma_i(p_i)\gamma_j(p_j)\phi_{\omega^f,\bar{\omega}^f}\left(\rho_i(p_i), \rho_j(p_j)\right),$$

$$\leq 0, \text{ by Lemma XIII.}$$ □

V.3 Condition (b) when $\lim_{p_j \to \infty} h_j(p_j) \geq 0$

In this section, we extend condition (b) in Theorem II to cases where $\lim_{p_j \to \infty} h_j(p_j)$ is not necessarily equal to zero. We start with the following technical lemma:

Lemma XIV. *Suppose that Assumption 1 holds for firm f, and that $\bar{\mu}_j = \bar{\mu}^f$ and ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ for every $j \in f$. Then, for every $k \in f$,*

$$S_k = \left\{ \omega \in (0, \bar{\omega}^f) : \ \exists p_k > \underline{p}_k, \ \omega = \chi_k(p_k) = \frac{1}{\rho_k(p_k)} \right\}$$

contains at most one element. If S_k is empty, then, either $\chi_k(p_k)\rho_k(p_k) > 1$ for every $p_k > \underline{p}_k$, or $\chi_k(p_k)\rho_k(p_k) < 1$ for every $p_k > \underline{p}_k$. If, instead, $S_k = \{\hat{\omega}\}$, then, for every $p_k > \underline{p}_k$,

- *$\theta_k(p_k) \leq \frac{1}{\hat{\omega}}$, and*
- *if $\rho_k(p_k) < \frac{1}{\hat{\omega}}$, then $\rho_k(p_k) \geq \frac{1-\hat{\omega}}{\hat{\omega}} \frac{1}{1-\chi_k(p_k)}$.*

Proof. Let $k \in f$, and assume for a contradiction that S_k contains two distinct elements. There exist $p_k, p'_k > \underline{p}_k$ such that $\chi_k(p_k)\rho_k(p_k) = 1$, $\chi_k(p'_k)\rho_k(p'_k) = 1$ and $\chi_k(p_k) \neq \chi_k(p'_k)$. To fix ideas, assume $\chi_k(p'_k) > \chi_k(p_k)$. Then, since χ_k is non-decreasing, $p'_k > p_k$. Since ρ_k is non-decreasing, $\rho_k(p_k) \leq \rho_k(p'_k)$. Therefore, $\chi_k(p_k)\rho_k(p_k) < \chi_k(p'_k)\rho_k(p'_k) = 1$, which is a contradiction.

Let $\kappa : p_k \in (\underline{p}_k, \infty) \mapsto \rho_k(p_k)\chi_k(p_k)$, and notice that κ is continuous and non-decreasing. If $S_k = \emptyset$, then, there is no p_k such that $\kappa(p_k) = 1$. Since κ is continuous, either $\kappa > 1$, or $\kappa < 1$.

Next, let $p_k > \underline{p}_k$. If $\rho_k(p_k) \leq \frac{1}{\hat{\omega}}$, then, $\theta_k(p_k) \leq \rho_k(p_k) \leq \frac{1}{\hat{\omega}}$. Assume instead that $\rho_k(p_k) > \frac{1}{\hat{\omega}}$. Let $\hat{p}_k$ such that $\chi_k(\hat{p}_k) = \hat{\omega} = \frac{1}{\rho_k(\hat{p}_k)}$. Then, $\rho_k(p_k) > \rho_k(\hat{p}_k) = \frac{1}{\hat{\omega}}$ and, by monotonicity, $p_k > \hat{p}_k$. Therefore, $\chi_k(p_k) \geq \chi_k(\hat{p}_k) = \hat{\omega}$. Moreover, by Lemma X, we have that $\theta_k(x) \leq \frac{1}{\chi_k(x)}$. It follows that $\theta_k(x) \leq \frac{1}{\chi_k(x)} \leq \frac{1}{\hat{\omega}}$.

Finally, assume that $\rho_k(p_k) < \frac{1}{\hat{\omega}}$. As in the previous paragraph, let $\hat{p}_k$ such that $\chi_k(\hat{p}_k) = \hat{\omega} = \frac{1}{\rho_k(\hat{p}_k)}$. By monotonicity, $\hat{p}_k > p_k$. Moreover, as already argued in the proof of Lemma XII, for every $t \in [p_k, \hat{p}_k]$,

$$\frac{\rho'_k(t)}{\rho_k(t)} = \frac{\iota'_k(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)} \left(\rho_k(t)\chi_k(t) - 1\right),$$

$$\leq \frac{\iota'_k(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)} \left(\rho_k(\hat{p}_k)\chi_k(\hat{p}_k) - 1\right), \text{ by monotonicity,}$$

$$= \frac{\iota_k'(t)}{\iota_k(t)}, \text{ by definition of } \hat{p}_k.$$

Integrating this inequality between p_k and $\hat{p}_k$, we obtain that $\frac{\rho_k(\hat{p}_k)}{\rho_k(p_k)} \leq \frac{\iota_k(\hat{p}_k)}{\iota_k(p_k)}$. Therefore,

$$\rho_k(p_k) \geq \rho_k(\hat{p}_k)\frac{\iota_k(p_k)}{\iota_k(\hat{p}_k)} = \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(p_k)}. \qquad \square$$

Proposition IV. *Suppose Assumption 1 holds for firm f. Assume that $\bar{\mu}^f = \bar{\mu}_j \leq \mu^*$, and that ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ for every $j \in f$. Assume also, using the notation introduced in Lemma XIV, that there exists $\hat{\omega} > 0$ such that, for every $j \in f$, $S_j = \{\hat{\omega}\}$. Then, s^f is strictly increasing.*

Proof. As in the proof of Lemma IX, the expression in condition (xiv) can be split in two terms (see equation (xv)). Since ρ_j is non-decreasing for every $j \in f$ and by Lemma X, the second sum is strictly negative. Next, we turn our attention to the first sum. Let $\omega^f \in (0, \bar{\omega}^f)$, $i, j \in f$, and p_i, p_j such that $\chi_i(p_i) > \omega^f$ and $\chi_j(p_j) > \omega^f$. We want to show that

$$\Psi = \omega^f\theta_i(p_i)\frac{1-\omega^f\rho_j(p_j)}{1-\omega^f\theta_i(p_i)} + \omega^f\theta_j(p_j)\frac{1-\omega^f\rho_i(p_i)}{1-\omega^f\theta_j(p_j)} - \rho_i(p_i) - \rho_j(p_j) \leq 0. \qquad \text{(xvi)}$$

To fix ideas, assume that $\rho_i(p_i) \leq \rho_j(p_j)$. If $\rho_i(p_i) \geq \frac{1}{\omega^f}$, then condition (xvi) is clearly satisfied, since, by Lemma X, $1-\omega^f\theta_i(p_i)$ and $1-\omega^f\theta_j(p_i)$ are strictly positive. Assume instead that $\rho_i(p_i) < \frac{1}{\omega^f}$. Then, we claim that $\omega^f < \hat{\omega}$. Assume for a contradiction that $\hat{\omega} \leq \omega^f$. Since $S_i = \{\hat{\omega}\}$, there exists $\hat{p}_i > \underline{p}_i$ such that $\chi_i(\hat{p}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{p}_i)}$. Therefore, $\rho_i(p_i) < \rho_i(\hat{p}_i)$ and, by monotonicity, $p_i < \hat{p}_i$. Since χ_i is non-decreasing, it follows that

$$\omega^f < \chi_i(p_i) \leq \chi_i(\hat{p}_i) = \hat{\omega},$$

which is a contradiction. Therefore, $\omega^f < \hat{\omega}$.

We distinguish three cases. Assume first that $\rho_j(p_j) < \frac{1}{\hat{\omega}}$. Then, by Lemma XIV,

$$\rho_k(p_k) \geq \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(p_k)} \geq \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\omega^f},$$

for $k \in \{i, j\}$. In addition, $\frac{\theta_i(p_i)}{1-\omega^f\theta_i(p_i)} \leq \frac{\rho_i(p_i)}{1-\omega^f\rho_i(p_i)}$ and $\frac{\theta_j(p_j)}{1-\omega^f\theta_j(p_j)} \leq \frac{\rho_j(p_j)}{1-\omega^f\rho_j(p_j)}$. Therefore,

$$\Psi \leq \phi_{\omega^f,\hat{\omega}}\left(\rho_i(p_i), \rho_j(p_j)\right),$$

which, by Lemma XIII, is non-positive, since $\hat{\omega} < \bar{\omega}^f \leq \omega^*$.

Next, assume that $\rho_i(p_i) < \frac{1}{\hat{\omega}} \leq \rho_j(p_j)$. Then, by Lemma XIV,

$$\rho_i(p_i) \geq \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_i(p_i)} \geq \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\omega^f},$$

and $\theta_j(p_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$
\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(p_i)}{1-\omega^f \theta_i(p_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}(1-\omega^f \rho_i(p_i)) - \rho_i(p_i) - \frac{1}{\hat{\omega}}, \\
&\leq \frac{\omega^f \rho_i(p_i)}{1-\omega^f \rho_i(p_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}(1-\omega^f \rho_i(p_i)) - \rho_i(p_i) - \frac{1}{\hat{\omega}}, \\
&= \phi_{\omega^f,\hat{\omega}}\left(\rho_i(p_i), \frac{1}{\hat{\omega}}\right), \\
&\leq 0 \text{ by Lemma XIII.}
\end{aligned}
$$

Finally, assume that $\rho_i(p_i) \geq \frac{1}{\hat{\omega}}$. By Lemma XIV, $\theta_i(p_i) \leq \frac{1}{\hat{\omega}}$ and $\theta_j(p_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$
\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(p_i)}{1-\omega^f \theta_i(p_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\omega^f \theta_j(p_j)}{1-\omega^f \theta_j(p_j)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}}, \\
&\leq \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}}, \\
&= \phi_{\omega^f,\hat{\omega}}\left(\frac{1}{\hat{\omega}}, \frac{1}{\hat{\omega}}\right), \\
&\leq 0, \text{ by Lemma XIII.}
\end{aligned}
$$

□

Condition $S_i = \{\hat{\omega}\}\ \forall i$ in Proposition IV may look a little bit arcane. The following corollary is easier to understand:

Corollary II. *Suppose firm f is such that there exist a $\mathcal{C}^3$, strictly decreasing and log-convex function h^f and strictly positive scalars $(\alpha_j, \beta_j)_{j\in f}$ such that $h_j(p_j) = \alpha_j h^f(\beta_j p_j)$ for every $j \in f$ and $p_j > 0$. Assume that h^f satisfies Assumption 1, ρ^f is non-decreasing on $(\underline{p}^f, \infty)$, and $\bar{\mu}^f < \mu^*$. Then, s^f is strictly increasing.*

Proof. It is straightforward to check that h_j satisfies Assumption 1, ρ_j is non-decreasing on $(\underline{p}_j, \infty)$, and $\bar{\mu}^f = \bar{\mu}_j$ for every $j \in f$. Next, we show that $S_i \subseteq S_j$ for all $i, j \in f$. Let $i, j \in f$. If S_i is empty, then, trivially, $S_i \subseteq S_j$. Assume instead that $S_i \neq \emptyset$, and let $\hat{\omega} \in S_i$. There exists $\hat{p}_i > \underline{p}_i$ such that

$$\chi_i(\hat{p}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{p}_i)}.$$

Since $h_i(p_i) = \alpha_i h^f(\beta_i p_i)$, it is easy to show that $\rho_i(\hat{p}_i) = \rho^f(\beta_i \hat{p}_i)$ and $\chi_i(\hat{p}_i) = \chi^f(\beta_i \hat{p}_i)$. Let $\hat{p}_j = \frac{\beta_i}{\beta_j}\hat{p}_i$. Then,

$$\chi_j(\hat{p}_j) = \chi^f\left(\beta_j \frac{\beta_i}{\beta_j}\hat{p}_i\right) = \chi_i(\hat{p}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{p}_i)} = \frac{1}{\rho^f(\beta_i \hat{p}_i)} = \frac{1}{\rho_j(\hat{p}_j)}.$$

Therefore, $\hat{\omega} \in S_j$, and $S_i \subseteq S_j$. It follows that $S_i = S_j$ for all $i, j \in f$.

If $S_i \neq \emptyset$, then, by Proposition IV, s^f is strictly increasing. Assume instead that $S_i = \emptyset$ for all i. Let $i \in f$. By Lemma XIV, either $\chi_i(p_i)\rho_i(p_i) < 1$ for all p_i, or $\chi_i(p_i)\rho_i(p_i) > 1$ for all p_i. Assume first that $\chi_i(p_i)\rho_i(p_i) < 1$ for all p_i. Let $j \in f$ and $p_j > \underline{p}_j$. Then,

$$\chi_j(p_j)\rho_j(p_j) = \chi_i\left(\frac{\beta_j}{\beta_i}p_j\right)\rho_i\left(\frac{\beta_j}{\beta_i}p_j\right) < 1.$$

Therefore, $\chi_j\rho_j < 1$ for every j in f. It follows that

$$\lim_{p_j\to\infty}\rho_j(p_j) \leq \lim_{p_j\to\infty}\frac{1}{\chi_j(p_j)} = \frac{1}{\bar{\omega}^f} < \infty.$$

Therefore, $\lim_{p_j\to\infty} h_j(p_j) = 0$ for every $j \in f$. (If $\lim_{p_j\to\infty} h_j(p_j)$ were strictly positive, then, since $\lim_{p_j\to\infty}\gamma_j(p_j) = 0$, $\rho_j(p_j)$ would go to ∞ as p_j goes to ∞.) Hence, by Lemma IX, s^f is strictly increasing.

Finally, assume that $\chi_i(p_i)\rho_i(p_i) > 1$ for all p_i. Then, using the same argument as above, $\chi_j\rho_j > 1$ for every $j \in f$. Let $i \in f$, and assume for a contradiction that $\underline{p}_i > 0$. Since $1/\chi_i$ is non-increasing, and since, by continuity, $\iota_i(\underline{p}_i) = 1$, it follows that $\lim_{p_i\to\underline{p}_i^+}\chi_i(p_i) = 0$. Therefore, $\lim_{p_i\to\underline{p}_i^+}\rho_i(p_i) = \infty$, which is a contradiction, since ρ_i is non-decreasing. Therefore, $\underline{p}_i = 0$.

Assume for a contradiction that $\lim_{p_i\to 0^+}\iota_i(p_i) = 1$. Then, using the same reasoning as in the previous paragraph, $\lim_{p_i\to 0^+}\rho_i(p_i) = \infty$, which is again a contradiction, since ρ_i is non-decreasing. Therefore, $\lim_{p_i\to 0^+}\iota_i(p_i) > 1$, and $\hat{\omega} \equiv \lim_{p_i\to 0^+}\chi_i(p_i)$ is strictly positive. In addition, since

$$\chi_j(p_j) = \chi_i\left(\frac{\beta_j}{\beta_i}p_j\right),$$

$\lim_{p_j\to 0^+}\chi_j(p_j) = \hat{\omega}$ for every $j \in f$. Notice that, for every $j \in f$, for every $p_j > 0$,

$$\rho_j(p_j) \geq \lim_{p'_j\to 0^+}\rho_j(p'_j) \geq \lim_{p'_j\to 0^+}\frac{1}{\chi_j(p'_j)} = \frac{1}{\hat{\omega}},$$

and that, by Lemma X,

$$\theta_j(p_j) \leq \frac{1}{\chi_j(p_j)} \leq \lim_{p'_j\to 0^+}\frac{1}{\chi_j(p'_j)} = \frac{1}{\hat{\omega}}.$$

It follows that

$$\max_{i\in f}\sup_{(0,\infty)}\theta_i \leq \frac{1}{\hat{\omega}} \leq \min_{i\in f}\inf_{(0,\infty)}\rho_i,$$

i.e., condition (a) in Theorem II holds. By Lemma VII, s^f is therefore strictly increasing. □

Proposition V. *Suppose Assumption 1 holds for firm f. Assume that $\bar{\mu}^f = \bar{\mu}_j \leq \mu^*$, and that ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ for every $j \in f$. Assume also that $\theta_j(p_j) \leq \frac{1}{\bar{\omega}^f}$ for every $j \in f$ and $p_j \in (\underline{p}_j, \infty)$. Then, s^f is strictly increasing.*

Proof. Let $i, j \in f$, $\omega^f \in (0, \bar{\omega}^f)$ and $p_i, p_j > 0$ such that $\chi_i(p_i) > \omega^f$ and $\chi_j(p_j) > \omega^f$. Define

$$\Psi = \frac{\omega^f \theta_i(p_i)}{1 - \omega^f \theta_i(p_i)} \left(1 - \omega^f \rho_j(p_j)\right) + \frac{\omega^f \theta_j(p_j)}{1 - \omega^f \theta_j(p_j)} \left(1 - \omega^f \rho_i(p_i)\right) - \rho_i(p_i) - \rho_j(p_j).$$

As in the previous proofs, all we need to do is show that $\Psi \leq 0$. Assume first that $\rho_i(p_i) \geq \frac{1}{\bar{\omega}^f}$ and $\rho_j(p_j) \geq \frac{1}{\bar{\omega}^f}$. Then,

$$\max\left(\theta_i(p_i), \theta_j(p_j)\right) \leq \min\left(\rho_i(p_i), \rho_j(p_j)\right).$$

Therefore, $\Psi < 0$.

Next, assume that $\rho_i(p_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(p_j) \geq \frac{1}{\bar{\omega}^f}$. Then, we claim that

$$\rho_i(p_i) \geq \frac{1 - \bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1 - \omega^f}. \tag{xvii}$$

To see this, assume first that $S_i = \{\hat{\omega}_i\}$, where $\hat{\omega}_i \in (0, \bar{\omega}^f)$. Since $\rho_i(p_i) < \frac{1}{\bar{\omega}^f} < \frac{1}{\hat{\omega}^i}$, by Lemma XIV,

$$\rho_i(p_i) \geq \frac{1 - \hat{\omega}_i}{\hat{\omega}_i} \frac{1}{1 - \chi_i(p_i)} \geq \frac{1 - \bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1 - \omega^f}.$$

Assume instead that $S_i = \emptyset$. By Lemma XIV, either $\chi_i \rho_i < 1$ or $\chi_i \rho_i > 1$. If $\chi_i \rho_i > 1$, then we know from the proof of Corollary II that

$$\rho_i \geq \sup \frac{1}{\chi_i} \geq \frac{1}{\bar{\omega}^f}.$$

This contradicts our assumption that $\rho_i(p_i) < \frac{1}{\bar{\omega}^f}$. If, instead, $\chi_i \rho_i < 1$, then we know from the proof of Corollary II that $\lim_{p_i' \to \infty} h_i(p_i') = 0$. Therefore, by Lemma XII, inequality (xvii) holds.

Therefore,

$$\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(p_i)}{1 - \omega^f \theta_i(p_i)} \left(1 - \frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1 - \frac{\omega^f}{\bar{\omega}^f}} \left(1 - \omega^f \rho_i(p_i)\right) - \rho_i(p_i) - \frac{1}{\bar{\omega}^f}, \\
&\leq \frac{\omega^f \rho_i(p_i)}{1 - \omega^f \rho_i(p_i)} \left(1 - \frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1 - \frac{\omega^f}{\bar{\omega}^f}} \left(1 - \omega^f \rho_i(p_i)\right) - \rho_i(p_i) - \frac{1}{\bar{\omega}^f}, \\
&= \phi_{\omega^f, \bar{\omega}^f}\left(\rho_i(p_i), \frac{1}{\bar{\omega}^f}\right), \\
&\leq 0 \text{ by Lemma XIII.}
\end{aligned}$$

Finally, assume that $\rho_i(p_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(p_j) < \frac{1}{\bar{\omega}^f}$. Then, as above,

$$\rho_k(p_k) \geq \frac{1 - \bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1 - \omega^f}$$

for $k \in \{i, j\}$. Therefore,

$$\Psi \leq \phi_{\omega^f, \bar{\omega}^f}\left(\rho_i(p_i), \rho_j(p_j)\right),$$

which is non-positive by Lemma XIII. □

Corollary III. *Suppose Assumption 1 holds for firm f. Assume that $\bar{\mu}^f = \bar{\mu}_j \leq \mu^*$, and that ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ for every $j \in f$. Assume also that θ_j is non-decreasing on $(\underline{p}_j, \infty)$ for every j in f. Then, s^f is strictly increasing.*

Proof. Let $k \in f$. Since θ_k is non-increasing, for every $p_k > \underline{p}_k$,

$$\theta_k(p_k) \leq \lim_{p'_k \to \infty} \theta_k(p'_k) \leq \lim_{p'_k \to \infty} \frac{1}{\chi_k(p'_k)} = \frac{1}{\bar{\omega}^f},$$

where the second inequality follows from Lemma X. Therefore, by Proposition V, s^f is strictly increasing. □

V.4 Proof of Proposition III

Proof. Let $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying Assumption 1. We rewrite the function Ω as follows:

$$\begin{aligned}\Omega(H) &= \sum_{f\in\mathcal{F}} \frac{\sum_{j\in f}\left(h_j(r_j(m^f(H))) + \frac{H^0}{|\mathcal{N}|}\right)}{H}, \\ &= \sum_{f\in\mathcal{F}} \frac{m^f(H)-1}{m^f(H)} \frac{\sum_{j\in f}\left(h_j(r_j(m^f(H))) + \frac{H^0}{|\mathcal{N}|}\right)}{\sum_{j\in f}\gamma_j(r_j(m^f(H)))},\end{aligned}$$

where we have used equation (14) in the paper. Hence, to establish equilibrium uniqueness, it is sufficient to show that, for every $f \in \mathcal{F}$, the ratio $\frac{\sum_{j\in f}\left(h_j(r_j(m^f(H)))+\frac{H^0}{|\mathcal{N}|}\right)}{\sum_{j\in f}\gamma_j(r_j(m^f(H)))}$ is strictly decreasing in H. This is equivalent to showing that the ratio $\xi^f(\mu^f) \equiv \frac{\sum_{j\in f}\left(h_j(r_j(\mu^f))+\frac{H^0}{|\mathcal{N}|}\right)}{\sum_{j\in f}\gamma_j(r_j(\mu^f))}$ is strictly increasing in μ^f.

Note that ξ^f is continuous on $(1, \bar{\mu}^f)$, and $\mathcal{C}^1$ on $(1, \bar{\mu}^f) \setminus \{\bar{\mu}_j\}_{j\in f}$. Hence, a sufficient condition for ξ^f to be strictly increasing is that $\xi^{f\prime}(\mu^f) > 0$ for every $\mu^f \in (1, \bar{\mu}^f) \setminus \{\bar{\mu}_j\}_{j\in f}$. Fix such a μ^f, and let f' be the set of j's such that $\bar{\mu}_j > \mu^f$. Then,

$$\xi^f(\mu^f) = \frac{\sum_{j\notin f'}\left(h_j(\infty) + \frac{H^0}{|\mathcal{N}|}\right) + \sum_{j\in f'}\left(h_j(r_j(\mu^f)) + \frac{H^0}{|\mathcal{N}|}\right)}{\sum_{j\in f'}\gamma_j(r_j(\mu^f))},$$

and

$$\xi^{f\prime}(\mu^f) = \frac{1}{\left(\sum_{j\in f'} \gamma_j(r_j(\mu^f))\right)^2} \left(-\sum_{j\notin f'} \left(h_j(\infty) + \frac{H^0}{|\mathcal{N}|}\right) \sum_{k\in f} r_k'(\mu^f)\gamma_k'(r_k(\mu^f)) \right.$$

$$\left. + \sum_{j,k\in f'} r_j'(\mu^f) \left(h_j'(r_j(\mu^f))\gamma_k(r_k(\mu^f)) - \gamma_j'(r_j(\mu^f)) \left(h_k(r_k(\mu^f)) + \frac{H^0}{|\mathcal{N}|} \right) \right) \right),$$

$$> \frac{\sum_{j,k\in f'} r_j'(\mu^f) \left(h_j'(r_j(\mu^f))\gamma_k(r_k(\mu^f)) - \gamma_j'(r_j(\mu^f)) \left(h_k(r_k(\mu^f)) + \frac{H^0}{|\mathcal{N}|} \right) \right)}{\left(\sum_{j\in f'} \gamma_j(r_j(\mu^f))\right)^2},$$

$$= \frac{\sum_{j,k\in f'} r_j'(\mu^f)(-\gamma_j'(r_j(\mu^f))) \left(-\theta_j(r_j(\mu^f))\gamma_k(r_k(\mu^f)) + h_k(r_k(\mu^f)) + \frac{H^0}{|\mathcal{N}|} \right)}{\left(\sum_{j\in f'} \gamma_j(r_j(\mu^f))\right)^2},$$

$$> \frac{\sum_{j,k\in f'} r_j'(\mu^f)(-\gamma_j'(r_j(\mu^f))) \left(-\theta_j(r_j(\mu^f))\gamma_k(r_k(\mu^f)) + \frac{H^0}{|\mathcal{N}|} \right)}{\left(\sum_{j\in f'} \gamma_j(r_j(\mu^f))\right)^2},$$

$$\geq \frac{\sum_{j,k\in f'} r_j'(\mu^f)(-\gamma_j'(r_j(\mu^f))) \left(-\frac{\gamma_k(r_k(\mu^f))}{\chi_j(r_j(\mu^f))} + \frac{H^0}{|\mathcal{N}|} \right)}{\left(\sum_{j\in f'} \gamma_j(r_j(\mu^f))\right)^2}, \qquad \text{(xviii)}$$

where the last inequality follows by Lemma X.

We can now prove the first part of the proposition. Let $\underline{H}^0 > 0$. Put $\underline{p} = \max_{f\in\mathcal{F}} \max_{j\in f} \underline{p}_j$. By Lemma A and Assumption 1, the functions $\gamma_k(\cdot)$ and $1/\chi_j(\cdot)$ are non-increasing on $(\underline{p}, \infty)$. Moreover, $\lim_{p_k\to\infty} \gamma_k(p_k) = 0$ and $\lim_{p_j\to\infty} 1/\chi_j(p_j) \geq 0$. Hence, there exists $\underline{c} > \underline{p}$ such that $\gamma_k(\underline{c})/\chi_j(\underline{c}) < \underline{H}^0/|\mathcal{N}|$ for every $f \in \mathcal{F}$ and $j,k \in f$. Suppose that the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ is such that $H^0 \geq \underline{H}^0$ and $c_i \geq \underline{c}$ for every $i \in \mathcal{N}$. Then, for every $i \in \mathcal{N}$ and $\mu \in (1, \bar{\mu}_i)$, we have that $r_i(\mu) \geq \underline{c}$. Hence, by monotonicity, for every $f \in \mathcal{F}$, $\mu^f \in (1, \bar{\mu}^f)$, and $j,k \in f$ such that $\bar{\mu}_j > \mu^f$ and $\bar{\mu}_k > \mu^f$, $\frac{\gamma_k(r_k(\mu^f))}{\chi_j(r_j(\mu^f))} < \frac{H^0}{|\mathcal{N}|}$. Using inequality (xviii), this implies that, for every firm f, for every μ^f, $\xi^{f\prime}(\mu^f) > 0$ whenever $\mu^f \notin \{\bar{\mu}_j\}_{j\in f}$. Hence, Ω is strictly decreasing, and the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium.

We now turn our attention to the second part of the lemma. Let $\underline{c} > 0$. For every $i \in \mathcal{N}$, let $\hat{p}_i$ be the monopolistic competition price for product i given marginal cost $\underline{c}$. Choose some $\underline{H}^0$ such that $\frac{\underline{H}^0}{|\mathcal{N}|} > \frac{\gamma_k(\hat{p}_k)}{\chi_j(\hat{p}_j)}$ for every $f \in \mathcal{F}$ and $j,k \in f$. (Since $\hat{p}_i > \underline{p}_i$ for every i, the ratios are well-defined.) Let $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying $H^0 \geq \underline{H}^0$ and $c_i \geq \underline{c}$ for every $i \in \mathcal{N}$. Since $c_i \geq \underline{c}$ for every i, we have that $r_i(\mu) \geq \hat{p}_i$ for every i. By monotonicity of the γ and χ functions, it follows that $\frac{\gamma_k(r_k(\mu^f))}{\chi_j(r_j(\mu^f))} < \frac{H^0}{|\mathcal{N}|}$ for every $f \in \mathcal{F}$, $\mu^f \in (1, \bar{\mu}^f)$, and $j,k \in f$ such that $\bar{\mu}_j > \mu^f$ and $\bar{\mu}_k > \mu^f$. Combining this with inequality (xviii) allows us to conclude that Ω is strictly decreasing, and that the pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium. $\square$

V.5 An Index Approach to Equilibrium Uniqueness

Fix a pricing game $((h_j)_{j\in\mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ satisfying Assumption 1. We now follow an index approach to derive conditions for equilibrium uniqueness. Since we will be working with matrices, we will sometimes assume that $\mathcal{F} = \{1, \ldots, F\}$, and that firm f's set of products is $\mathcal{N}^f$. To avoid differentiability issues which would prevent us from applying the index theorem, we assume that $\bar{\mu}_j = \bar{\mu}^f$ for every $f \in \mathcal{F}$ and $j \in f$.

We know that establishing uniqueness in the pricing game is equivalent to establishing uniqueness in the auxiliary game in which firms are simultaneously choosing their μ^f's. We also know that a profile $\mu = (\mu^f)_{f\in\mathcal{F}}$ is an equilibrium of the auxiliary game if and only if for every $f \in \mathcal{F}$,

$$\phi^f(\mu) \equiv (\mu^f - 1)\left(\left(\sum_{k\in\mathcal{N}^f} h_k\right) + \left(\sum_{\substack{g\in\mathcal{F}\\ g\neq f}} \sum_{k\in\mathcal{N}^f} h_k\right) + H^0\right) - \mu^f \sum_{k\in\mathcal{N}^f} \gamma_k = 0.$$

In the following, we derive conditions under which the map ϕ has a unique zero. We do so by showing that, under those conditions, the determinant of the Jacobian matrix of ϕ evaluated at μ is strictly positive whenever $\phi(\mu) = 0$. We have shown in the proof of Lemma G that

$$\frac{\partial \phi^f}{\partial \mu^f} = \sum_{f\in\mathcal{F}} \sum_{k\in\mathcal{N}^f} h_k + H^0 \equiv H(\mu).$$

Moreover, if $g \neq f$, then

$$\frac{\partial \phi^f}{\partial \mu^g} = (\mu^f - 1) \sum_{k\in\mathcal{N}^g} r'_k h'_k.$$

Therefore,

$$\begin{aligned}
\det J(\phi) &= \begin{vmatrix} H(\mu) & (\mu_1 - 1)\sum_{k\in\mathcal{N}^2} r'_k h'_k & \cdots & (\mu_1 - 1)\sum_{k\in\mathcal{N}^F} r'_k h'_k \\ (\mu_2 - 1)\sum_{k\in\mathcal{N}^1} r'_k h'_k & H(\mu) & \cdots & (\mu_2 - 1)\sum_{k\in\mathcal{N}^F} r'_k h'_k \\ \vdots & \vdots & \ddots & \vdots \\ (\mu^F - 1)\sum_{k\in\mathcal{N}^1} r'_k h'_k & (\mu^F - 1)\sum_{k\in\mathcal{N}^2} r'_k h'_k & \cdots & H(\mu) \end{vmatrix}, \\
&= \left(\prod_{f\in\mathcal{F}} (\mu^f - 1) \sum_{k\in\mathcal{N}^f} r'_k h'_k\right) \det \mathcal{M}\left(\left(1 + \frac{H(\mu)}{(\mu^f - 1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)_{1\leq f\leq F}\right),
\end{aligned}$$

where the second line has been obtained by dividing row f by $\mu^f - 1$ and column f by $\sum_{k\in\mathcal{N}^f} r'_k h'_k$ for every f in $\{1, \ldots, F\}$, and by using the F-linearity of the determinant. By

Lemma I,

$$
\begin{aligned}
\det\left(J(\phi)\right) &= \left(\prod_{f\in\mathcal{F}}(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k h'_k\right)(-1)^F \left(\left(\prod_{f\in\mathcal{F}}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right)\right. \\
&\quad \left. -\sum_{g\in\mathcal{F}}\prod_{f\neq g}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right), \\
&= \left(\prod_{f\in\mathcal{F}}(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k h'_k\right)(-1)^F \left(\prod_{f\in\mathcal{F}}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right) \\
&\quad \times\left(1-\sum_{f\in\mathcal{F}}\frac{1}{1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}}\right), \\
&= \underbrace{\left(\prod_{f\in\mathcal{F}}\left(H(\mu)+(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)\right)\right)}_{>0}\left(1-\sum_{f\in\mathcal{F}}\frac{1}{1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}}\right).
\end{aligned}
$$

Therefore, we need to show that

$$
\sum_{f\in\mathcal{F}}\frac{\frac{\mu^f-1}{H(\mu)}\sum_{k\in f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{H(\mu)}\sum_{k\in f} r'_k(-h'_k)} < 1 \tag{xix}
$$

whenever $\phi(\mu)=0$.

We now relate this uniqueness condition to the one we derived by following an aggregative games approach. Applying the implicit function theorem to equation (14), we obtain:

$$
\begin{aligned}
m^{f\prime}(H) &= \frac{-1}{H}\frac{m^f(H)(m^f(H)-1)}{1+m^f(H)(m^f(H)-1)\frac{\sum_{k\in f} r'_k(-\gamma'_k)}{\sum_{k\in f}\gamma_k}}, \\
&= \frac{-1}{H}\frac{m^f(H)(m^f(H)-1)}{1+(m^f(H)-1)\frac{\sum_{k\in f}\left((m^f(H)-1)r'_k(-h'_k)+\gamma_k\right)}{\sum_{k\in f}\gamma_k}}, \\
&= \frac{-1}{H}\frac{m^f(H)(m^f(H)-1)}{m^f(H)+(m^f(H)-1)^2\frac{\sum_{k\in f} r'_k(-h'_k)}{\sum_{k\in f}\gamma_k}}, \\
&= -\frac{\frac{m^f(H)-1}{H}}{1+\frac{m^f(H)-1}{H}\sum_{k\in f} r'_k(-h'_k)},
\end{aligned}
$$

where the second line follows by equation (11), and the fourth line follows by equation (14).

The derivative of the aggregate fitting-in function is therefore given by:

$$\Gamma'(H) = \sum_{f\in\mathcal{F}} m^{f\prime}(H) \sum_{k\in f} r'_k h'_k = \sum_{f\in\mathcal{F}} \frac{\frac{m^f-1}{H}\sum_{k\in f} r'_k(-h'_k)}{1+\frac{m^f-1}{H}\sum_{k\in f} r'_k(-h'_k)}.$$

The index condition (xix) is therefore equivalent to the fact that the slope of the aggregate fitting-in function is strictly less than unity whenever that function intersects the 45-degree line. This is, in turn, equivalent to $\Omega'(H) < 0$ whenever $\Omega(H) = 1$, which is an index condition for the mapping $\Omega - 1$.

VI Functional Forms and Cookbooks for Applied Work

VI.1 Equilibrium Existence: Functional Forms and Cookbook

Recall that $\mathcal{H}^\iota$ was defined as the set of $\mathcal{C}^3$, strictly decreasing and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$ that satisfy Assumption 1. In this section, we provide examples of functions h that belong to $\mathcal{H}^\iota$. We also develop a cookbook for constructing such functions.

Cookbook. One way of looking for an example of a function h that belongs to $\mathcal{H}^\iota$ is to start with a function h that is positive, decreasing and log-convex, and check that the associated ι function is non-decreasing whenever it is strictly greater than 1. This is tedious, because nothing guarantees that ι will have the right monotonicity property. Another possibility is to start with a function ι that is positive and non-decreasing, integrate a second-order differential equation to obtain a function h, and adjust constants of integration to ensure that h is positive, decreasing and log-convex. The following proposition states that such constants of integration exist:

Proposition VI. *Let $\widetilde{\iota}: \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ be a $\mathcal{C}^1$ function such that $\widetilde{\iota}$ is non-decreasing, $\lim_{p\to 0^+} \widetilde{\iota}(p) > 0$, and $\widetilde{\iota}(p) > 1$ for some $p > 0$. For every $(\alpha, \beta) \in \mathbb{R}^2_{++}$, let*

$$h^{\alpha,\beta}(p) = \alpha\left(\beta - \int_1^p \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u}du\right)dt\right).$$

Then, there exists $\underline{\beta} > 0$ such that $h^{\alpha,\beta} \in \mathcal{H}^\iota$ if and only if $\alpha > 0$ and $\beta \geq \underline{\beta}$.

Proof. It is straightforward to show, using standard differential equation techniques, that $-x\frac{h''(x)}{h'(x)} = \widetilde{\iota}(x)$ for all x if and only if $h = h^{\alpha,\beta}$ for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$. All we need to do now is look for the set of pairs (α, β) such that $h^{\alpha,\beta} \in \mathcal{H}^\iota$.

Note that, for all α, β,

$$h^{\alpha,\beta\prime}(x) = -\alpha \exp\left(-\int_1^x \frac{\iota(u)}{u}du\right),$$

i.e., $h^{\alpha,\beta\prime}$ has the same sign as $-\alpha$. It follows that $h^{\alpha,\beta}$ cannot be in $\mathcal{H}^\iota$ if $\alpha \leq 0$. In addition, if $h^{\alpha,\beta} \in \mathcal{H}^\iota$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$, then $h^{\alpha',\beta} \in \mathcal{H}^\iota$ for all $\alpha' > 0$. Therefore, we can set α equal to 1 without loss of generality.

The problem now boils down to finding the set of β's such that $h^\beta \equiv h^{1,\beta}$ is strictly positive, decreasing and log-convex. We already know that $h^{\beta\prime} < 0$. Therefore, the fact that h^β has to be decreasing does not impose any constraint on β.

Clearly, $\lim_{p\to\infty} h^0(p)$ exists and is strictly negative. We now show that this limit is finite. Let $x^0 > 0$ such that $\widetilde{\iota}(x^0) > 1$. Proving that $\lim_{p\to\infty} h^0(p)$ is finite is equivalent to showing that the function $t \mapsto \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u}du\right)$ is integrable on $[x^0, \infty)$. For every $t \geq x^0$,

$$\begin{aligned}
\exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u}du\right) &\leq \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u}du - \int_{x^0}^t \frac{\iota(x^0)}{u}du\right), \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u}du\right) \exp\left(-\widetilde{\iota}(x^0)\log\left(\frac{t}{x^0}\right)\right), \qquad \text{(xx)} \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u}du\right) \left(\frac{t}{x^0}\right)^{-\widetilde{\iota}(x^0)}.
\end{aligned}$$

The last expression is integrable on $[x^0, \infty)$, since $\widetilde{\iota}(x^0) > 1$. Therefore, $t \mapsto \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u}du\right)$ is integrable on $[x^0, \infty)$ and $\hat{\beta} \equiv \lim_{p\to\infty} h^0(p)$ is finite and strictly negative. It follows that the function h^β is strictly positive if and only if $\beta \geq \hat{\beta}$.

Let $\beta \geq \hat{\beta}$. Then,

$$\frac{d}{dx}\frac{h^{\beta\prime}(x)}{h^\beta(x)} = \frac{h^{\beta\prime\prime}(x)h^\beta(x) - \left(h^{\beta\prime}(x)\right)^2}{h^\beta(x)^2} = \frac{1}{x}\frac{-h^{\beta\prime}(x)}{h^\beta(x)}\left(\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}\right).$$

Therefore, h^β is log-convex if and only if $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x > 0$. Since $h^\beta(x)$ increases with β and $h^{\beta\prime}(x)$ does not depend on β, it follows that, if h^β is log-convex and $\beta' > \beta$, then $h^{\beta'}$ is also log-convex.

Moreover, using (xx), we see that, for every $x > x^0$,

$$\begin{aligned}
-xh^{\beta\prime}(x) &\leq x\exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u}du\right)\left(\frac{x}{x^0}\right)^{-\widetilde{\iota}(x^0)}, \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u}du\right)\left(x^0\right)^{\widetilde{\iota}(x^0)} x^{1-\widetilde{\iota}(x^0)} \underset{x\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where the last line follows from the fact that $\widetilde{\iota}(x^0) > 1$.

Let $\beta > \hat{\beta}$. Then, $\lim_{p\to\infty} h^\beta(p) > 0$, and therefore, $\lim_{x\to\infty} x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} = 0$. Since $\lim_{p\to\infty}\widetilde{\iota}(p) > 0$, it follows that there exists $\hat{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ whenever $x \geq \hat{x}$.

In addition, since h^β increases with β, we also have that, for all $\beta' \geq \beta$, $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)}$ whenever $x \geq \hat{x}$.

Next, we turn our attention to $\lim_{x\to 0^+} \frac{-xh^{\beta\prime}(x)}{h^\beta(x)}$. Note that

$$\frac{d}{dx}(-xh^{\beta\prime}(x)) = -h^{\beta\prime}(x)\left(1-\widetilde{\iota}(x)\right).$$

Therefore, if $\lim_{p\to 0^+}\widetilde{\iota}(p) > 1$ or $\lim_{p\to 0^+}\widetilde{\iota}(p) < 1$, then $x \mapsto (-xh^{\beta\prime}(x))$ is monotone in the neighborhood of zero, and $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ exists. If instead $\lim_{0^+}\widetilde{\iota} = 1$, then, by monotonicity, either there exists $\varepsilon > 0$ such that $\widetilde{\iota}(x) = 1$ for all $x \in (0,\varepsilon)$, or $\widetilde{\iota}(x) > 1$ for all $x > 0$. In both cases, $x \mapsto (-xh^{\beta\prime}(x))$ is monotone in the neighborhood of zero, and $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ therefore exists. Note that $\lim_{p\to 0^+} h^\beta(p)$ trivially exists, since h^β is monotone.

We distinguish two cases. Suppose first that $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ is finite, and denote this limit by l. If $\lim_{0^+} h^\beta = \infty$, then

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} \underset{x\to 0^+}{\longrightarrow} \lim_{p\to 0^+}\widetilde{\iota}(p) > 0.$$

Therefore, there exists $\tilde{x} > 0$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in (0,\tilde{x}]$. In addition, the inequality also holds if we replace β by $\beta' \geq \beta$. If, instead, $\lim_{0^+} h^\beta < \infty$, then

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} \underset{x\to 0^+}{\longrightarrow} \underbrace{\lim_{p\to 0^+}\widetilde{\iota}(p)}_{>0} - \frac{l}{\lim_{p\to 0^+} h^{\hat{\beta}}(p) + \beta - \hat{\beta}},$$

which is strictly positive for β high enough. For such a high enough β, we obtain again the existence of an $\tilde{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in (0,\tilde{x}]$.

Next, assume instead that $\lim_{x\to 0^+} -xh^{\beta\prime}(x) = \infty$. Let $M > 0$. There exists $\varepsilon > 0$ such that $h^{\beta\prime}(x) < -M/x$ whenever $x \leq \varepsilon$. Integrating this inequality between x and ε, we see that

$$h^\beta(x) > h^\beta(\varepsilon) + M\log\frac{\varepsilon}{x} \underset{x\to 0^+}{\longrightarrow} \infty.$$

Therefore, $\lim_{p\to 0^+} h^\beta(p) = \infty$, and we can apply l'Hospital's rule:

$$\lim_{x\to 0^+}\frac{-xh^{\beta\prime}(x)}{h^\beta(x)} = \lim_{x\to 0^+}\frac{-xh^{\beta\prime\prime}(x) - h^{\beta\prime}(x)}{h^{\beta\prime}(x)} = \lim_{p\to 0^+}\widetilde{\iota}(p) - 1.$$

Therefore,

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} \underset{x\to 0^+}{\longrightarrow} 1 > 0.$$

Again, this gives us the existence of an $\tilde{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in (0,\tilde{x}]$.

To summarize, we have found a $\beta > \hat{\beta}$ and two strictly positive reals $\tilde{x}$ and $\hat{x}$ such that for all $\beta' \geq \beta$, $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ whenever $x \geq \hat{x}$ or $x \leq \tilde{x}$. If $\tilde{x} \geq \hat{x}$, then we are done: there exists $\beta > \hat{\beta}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)}$ for all $x > 0$. Assume instead that $\tilde{x} < \hat{x}$. Then, for every $\beta' \geq \beta$ and $x \in [\tilde{x}, \hat{x}]$,

$$\begin{aligned}
x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)} &\leq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(\hat{x})}, \text{ since } h^{\beta'} \text{ is non-increasing,} \\
&= x\frac{-h^{\beta\prime}(x)}{h^{\beta}(\hat{x}) + \beta' - \beta}, \text{ since } h^{\beta'} - h^{\beta} = \beta' - \beta, \\
&\leq \underbrace{\max_{t \in [\tilde{x}, \hat{x}]} \left(-th^{\beta\prime}(t)\right)}_{\text{finite, by continuity and compactness}} \frac{1}{h^{\beta}(\hat{x}) + \beta' - \beta} \underset{\beta' \to \infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, there exists $\beta' \geq \beta$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ for all $x \in [\tilde{x}, \hat{x}]$. It follows that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ for all $x > 0$.

This implies that the set

$$B \equiv \left\{\beta \geq \hat{\beta} : \quad h^{\beta} \text{ is log-convex}\right\}$$

is non-empty. In addition, we also know that if $\beta' > \beta$ and $\beta \in B$, then $\beta' \in B$. Put $\underline{\beta} = \inf B$. Assume for a contradiction that $\underline{\beta} \notin B$. Then, there exists $x > 0$ such that

$$\widetilde{\iota}(x) < x\frac{-h^{\underline{\beta}\prime}(x)}{h^{\underline{\beta}}(x)}.$$

Then, by continuity of h^{β} in β, there exists $\beta' > \underline{\beta}$ such that

$$\widetilde{\iota}(x) < x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}.$$

But then, $\beta' \in B$ and $h^{\beta'}$ is not log-convex, a contradiction. Therefore, the set of β's such that h^{β} is positive, decreasing and log-convex is $[\underline{\beta}, \infty)$. □

The appeal of Proposition VI is that it allows us to use $(\iota_j)_{j \in \mathcal{N}}$ as a primitive, instead of $(h_j)_{j \in \mathcal{N}}$. This is useful, because markup patterns are governed by the ι functions.

Once an admissible h function has been generated, it is straightforward to modify it by introducing price sensitivity and quality parameters:

Proposition VII. *Let $h \in \mathcal{H}^{\iota}$ and $(\alpha, \beta, \delta, \epsilon) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{+}$. For every $p > 0$, define*

$$\tilde{h}(p) = \alpha h\left(\beta p + \delta\right) + \varepsilon.$$

Then, $\tilde{h} \in \mathcal{H}^{\iota}$.

Proof. See the proof of Proposition VIII. □

Examples. As we mention in the paper, the set of CES ($h_i(p_i) = a_i p_i^{1-\sigma}$, $a_i > 0$, $\sigma > 1$) and MNL ($h_i(p_i) = \exp((a_i - p_i)/\lambda_i)$, $a_i \in \mathbb{R}$, $\lambda_i > 0$) h-functions is contained in $\mathcal{H}^\iota$. One way of bridging the gap between CES and MNL functions is to consider the following family of h-functions: For every $\lambda > 0$, $\phi \in [0,1]$ and $p > 0$,

$$h^{\phi,\lambda}(p) = \begin{cases} \exp\left(-\lambda \frac{p^\phi - 1 + \phi^2}{\phi}\right) & \text{if } \phi > 0, \\ p^{-\lambda} & \text{if } \phi = 0. \end{cases}$$

It is easy to check that $h^{\phi,\lambda}$ converges pointwise to $h^{0,\lambda}$ (i.e., CES) when ϕ goes to zero, and to MNL when ϕ goes to 1, and that $h^{\phi,\lambda} \in \mathcal{H}^\iota$ for every ϕ, λ.

Other examples of admissible h-functions include $h(p) = 1/\log(1 + \mathrm{e}^p)$, $h(p) = \exp(\mathrm{e}^{-p})$, $h(p) = 1 + 1/(1 + \mathrm{e}^{1+p})$, $h(p) = 1 + 1/\cosh(2 + x)$, etc. All these functions can be tweaked by adding price sensitivity and quality parameters, as described in Proposition VII.

VI.2 Equilibrium Uniqueness: Functional Forms and Cookbook

Cookbook. A priori, condition (a) in Theorem II seems tedious to check if the firm under consideration has heterogeneous products. The following proposition shows that a certain type of product heterogeneity can be easily handled, and provides a cookbook for applied work:

Proposition VIII. *Let $h \in \mathcal{H}^\iota$ such that $\sup_{p > \underline{p}} \theta(p) \leq \inf_{p > \underline{p}} \rho(x)$. Let f be a finite and non-empty set, and, for every $j \in f$, $(\alpha_j, \beta_j, \delta_j, \epsilon_j) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_+$. For every $j \in f$, define*

$$h_j(p_j) = \alpha_j h(\beta_j p_j + \delta_j) + \epsilon_j, \quad \forall p_j > 0.$$

Then, for all $j \in f$, $h_j \in \mathcal{H}^\iota$. Moreover, $\max_{j \in f} \sup_{p_j > \underline{p}_j} \theta_j(p_j) \leq \min_{j \in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j)$.

Proof. Let $j \in f$. Then, for all $p > 0$,

$$\begin{aligned} h_j'(p) &= \alpha_j \beta_j h'(\beta_j p + \delta_j) < 0, \\ h_j''(p) &= \alpha_j \beta_j^2 h''(\beta_j p + \delta_j) > 0, \\ \gamma_j(p) &= \alpha_j \gamma(\beta_j p + \delta_j), \\ \gamma_j'(p) &= \alpha_j \beta_j \gamma'(\beta_j p + \delta_j), \\ \rho_j(p) &= \rho(\beta_j p + \delta_j) + \frac{\epsilon_j}{\alpha_j \gamma(\beta_j p + \delta_j)} \geq \rho(\beta_j p + \delta_j), \\ \theta_j(p) &= \theta(\beta_j p + \delta_j), \\ \iota_j(p) &= \frac{\beta_j p}{\beta_j p + \delta_j} \iota(\beta_j p + \delta_j). \end{aligned}$$

Therefore, h_j is positive, decreasing and log-convex, and ι_j is non-decreasing whenever ι_j is > 1. In addition, for every $p > \underline{p}_j$,

$$1 < \iota_j(p) \leq \iota(\beta_j p + \delta_j).$$

Therefore, $\beta_j p + \delta_j > \underline{p}$, and

$$\theta_j(p) \leq \sup_{p' > \underline{p}} \theta(p').$$

It follows that $\sup_{p > \underline{p}_j} \theta_j(p) \leq \sup_{p > \underline{p}} \theta(p)$. Using the same reasoning, we also obtain that $\inf_{p > \underline{p}_j} \rho_j(p) \geq \inf_{p > \underline{p}} \rho(p)$. Therefore,

$$\begin{aligned}
\max_{j \in f} \sup_{p > \underline{p}_j} \theta_j(p) &\leq \max_{j \in f} \sup_{p > \underline{p}} \theta(p), \\
&\leq \sup_{p > \underline{p}} \theta(p), \\
&\leq \inf_{p > \underline{p}} \rho(p), \\
&\leq \min_{j \in f} \inf_{p > \underline{p}} \rho(p), \\
&\leq \min_{j \in f} \inf_{p > \underline{p}_j} \rho_j(p).
\end{aligned}$$

□

Examples. Proposition VIII can be applied as follows. Let $h(p) = \mathrm{e}^{-p}$ for all $p > 0$. We already know that $h \in \mathcal{H}^\iota$. In addition, $\rho(p) = \theta(p) = 1$ for all $p > 0$. By Proposition VIII, if firm f is such that for all $j \in f$, there exist $\lambda_j > 0$ and $a_j \in \mathbb{R}$ such that $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda_j}}$ for all $p_j > 0$ (i.e., firm f only has MNL products), then condition (a) in Theorem II holds for firm f. This implies in particular that a multiproduct-firm pricing game with MNL demand has a unique equilibrium.

Similarly, let $h(p) = p^{1-\sigma}$ for all $p > 0$ ($\sigma > 1$). Again, we already know that $h \in \mathcal{H}^\iota$. In addition, $\rho(p) = \theta(p) = \sigma/(\sigma - 1)$. Therefore, if firm f is such that for all $j \in f$, there exist $a_j, b_j, d_j > 0$ such that $h_j(p_j) = a_j \left(b_j p_j + d_j\right)^{1-\sigma}$ for all $p_j > 0$, then condition (a) in Theorem II holds for firm f. In particular, a pricing game with CES demand has a unique equilibrium. Other candidates for the base h include $h(x) = \exp\left(\mathrm{e}^{-x}\right)$, $h(x) = 1 + 1/(1 + \mathrm{e}^{1+x})$, $h(x) = 1 + 1/\cosh(2 + x)$, etc.

Some functions satisfy condition (b) in Theorem II, but not condition (a). Consider the following function: $h(x) = \frac{1}{\log(1+\mathrm{e}^x)}$. It is easy to show that $h \in \mathcal{H}^\iota$, ρ is non-increasing, and $\bar{\mu} = 2 (< 2.78)$. Therefore, condition (b) holds. However, condition $\sup \theta(x) \leq \inf \rho(x)$ is not satisfied.

It is easy to find functional forms for which Theorem II has no bite. Consider, for instance, the family of functions $h^{\phi,\lambda} \in \mathcal{H}^\iota$ introduced in Section VI.1. It is easy to show that $\rho^{\phi,\lambda}(\cdot)$ is strictly decreasing whenever $\phi \in (0, 1)$. Therefore, none of the conditions in Theorem II hold. With such functional forms, it is still possible to apply Proposition III to prove uniqueness of equilibrium, provided that marginal costs are sufficiently high and/or that the outside option is sufficiently attractive.

VII Nested Demand Systems and Multi-Stage Discrete / Continuous Choice

VII.1 Multi-Stage Discrete/Continuous Choice

We model multi-stage discrete/continuous choice as follows. The (non-empty and finite) set of products $\mathcal{N}$ is partitioned into a set of nests $\mathcal{M}$. There is a continuum of consumers. The type of a consumer is denoted by $(\eta_0, \eta) \in [-\infty, \infty)^2$. These types are distributed according to the measure μ. A consumer of type (η_0, η) observes his type and the price vector $(p_j)_{j \in \mathcal{N}}$ at the beginning of the choice process. He first decides whether to take the outside option, in which case he receives the utility flow η_0, or to continue searching. If he turns down the outside option, then he receives the utility flow η, and moves on to the second stage of the choice process. He then observes a vector of nest-level taste shocks $(\varepsilon^m)_{m \in \mathcal{M}}$, drawn i.i.d. from a type-I extreme value distribution. If he picks nest $m \in \mathcal{M}$, then he receives the utility flow ε^m, and moves on to the third stage. In that third stage, he observes a random pair (η_0^m, η^m) drawn from a probability measure ν^m over $[-\infty, \infty)^2$, and decides whether or not to take the nest-specific outside option. If he does take that outside option, then he receives the utility flow η_0^m. If not, then he receives the utility flow η^m, and moves on to the fourth and last stage of the choice process. In that last stage, he observes a vector of product-level taste shocks $(\varepsilon_j)_{j \in m}$ drawn i.i.d. from a type-I extreme-value distribution, chooses a product $j \in m$, receives the utility flow $\log h_j(p_j) + \varepsilon_j$, and consumes $-h_j'(p_j)/h_j(p_j)$ units of that product. Consumers are assumed to be expected utility maximizers.

Thus, if a consumer of type (η_0, η) turns down the outside option in stage 1, chooses nest $m \in \mathcal{M}$ in stage 2, turns down the nest-specific outside option in stage 3, and chooses product $j \in m$, then that consumer receives the utility flow $\log h_j(p_j) + \varepsilon_j + \eta^m + \varepsilon^m + \eta$. If instead he turns down the outside option in stage 1, chooses nest $m \in \mathcal{M}$ in stage 2, but takes the outside option in stage 3, then he receives the utility flow $\eta_0^m + \varepsilon^m + \eta$.

To summarize, a multi-stage discrete/continuous choice process is a tuple $(\mathcal{N}, \mathcal{M}, \mu, (\nu^m)_{m \in \mathcal{M}}, (h_j)_{j \in \mathcal{N}})$, where $\mathcal{N}$ is a non-empty and finite set, $\mathcal{M}$ is a partition of $\mathcal{N}$, μ is a measure over $[-\infty, \infty)^2$, ν^m is a probability measure over $[-\infty, \infty)^2$ for every $m \in \mathcal{M}$, and h_j is a strictly positive and $\mathcal{C}^1$ function for every $j \in \mathcal{N}$. Throughout this section, we maintain the following assumption:

Assumption ii. *(a) For every $j \in \mathcal{N}$, h_j is a $\mathcal{C}^1$, strictly decreasing, and log-convex function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$.*

(b) For every $X \in \mathbb{R}$, the function $(\eta_0, \eta) \in [-\infty, \infty)^2 \mapsto \max(\eta_0, X + \eta)$ is μ-integrable,

and

$$\mu\left(\{(\eta_0,\eta)\in\mathbb{R}^2:\ \eta-\eta_0=X\}\right)=0.$$

(c) *For every $m\in\mathcal{M}$ and $X\in\mathbb{R}$, the function $(\eta_0^m,\eta^m)\in[-\infty,\infty)^2\mapsto\max(\eta_0^m,X+\eta^m)$ is ν^m-integrable, and the random variable $\eta^m-\eta_0^m$ (where (η_0^m,η^m) is drawn from the probability measure ν^m, conditionally on (η_0^m,η^m) being finite) is continuously distributed.*

As in Section I, Assumption ii-(a) ensures that $\log h_j$ is an indirect subutility function for every j. The integrability parts of Assumptions ii-(b) and (c) ensure that consumer surplus is well-defined. The atomless parts of Assumptions ii-(b) and (c) will give us smooth choice probabilities.

The following proposition provides a complete characterization of the set of demand systems that can be derived from multi-stage discrete/continuous choice.

Proposition IX. *Let $D:\mathbb{R}^{\mathcal{N}}_{++}\longrightarrow\mathbb{R}^{\mathcal{N}}_{+}$ be a demand system. The following assertions are equivalent:*

(i) *D can be derived from a model of multi-stage discrete/continuous choice $(\mathcal{N},\mathcal{M},\mu,(\nu^m)_{m\in\mathcal{M}},(h_j)_{j\in\mathcal{N}})$ satisfying Assumption ii.*

(ii) *There exist functions Ψ, $(\Phi^m)_{m\in\mathcal{M}}$ and $(h_j)_{j\in\mathcal{N}}$ such that, for every $p\in\mathbb{R}^{\mathcal{N}}_{++}$, $n\in\mathcal{M}$ and $i\in n$,*

$$D_i(p)=-h_i'(p_i)\Phi^{n\prime}\left(\sum_{j\in n}h_j(p_j)\right)\Psi'\left(\sum_{m\in\mathcal{M}}\Phi^m\left(\sum_{k\in m}h_k(p_k)\right)\right),\qquad\text{(xxi)}$$

where:

(a) *For every $i\in\mathcal{N}$, h_i is $\mathcal{C}^1$, strictly decreasing, and log-convex from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$,*

(b) *For every $n\in\mathcal{M}$, Φ^n is $\mathcal{C}^1$ from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$; Moreover, $H\mapsto\frac{H\Phi^{n\prime}(H)}{\Phi^n(H)}$ is non-negative, non-decreasing, and bounded above by 1,*

(c) *Ψ is $\mathcal{C}^1$ from $\mathbb{R}_{++}$ to $\mathbb{R}$; Moreover, $\Phi\mapsto\Phi\Psi'(\Phi)$ is non-negative and non-decreasing.*

Moreover, overall consumer surplus at price vector p is equal to $\Psi\left(\sum_{m\in\mathcal{M}}\Phi^m\left(\sum_{k\in m}h_k(p_k)\right)\right)$.

Proof. **(i) ⇒ (ii).** let $(\mathcal{N},\mathcal{M},\mu,(\nu^m)_{m\in\mathcal{M}},(h_j)_{j\in\mathcal{N}})$ be a model of multi-stage discrete/continuous choice satisfying Assumption ii. We fix a consumer, and compute his expected demand for each product. We know from our analysis in Section I that, if the consumer ends up in nest n in the fourth stage of the choice process, then he chooses product $i\in n$ with probability $h_i(p_i)/\sum_{j\in n}h_j(p_j)$, and consumes $-h_i'(p_i)/h_i(p_i)$ units of that product. Moreover, his expected utility from choosing nest n in stage 3 is $\log\sum_{j\in n}h_j(p_j)+\eta^n\equiv\log H^n+\eta^n$.

Hence, if the consumer ends up in nest n in the third stage of the choice process, then he turns down the nest-specific outside option if and only if $\log H^n+\eta^n\geq\eta_0^n$. (Ties are irrelevant, since, by Assumption ii-(c), the event $\eta_0^n - \eta^n = \log H^n$ arises with probability zero if η_0^n and η^n are both finite, and the integrability condition implies that the event $(\eta_0^n, \eta^n) = (-\infty, -\infty)$ is assigned probability zero as well.) Conditional on choosing nest n in stage 2, the consumer's expected utility (which is well defined, due to the integrability part of Assumption ii-(c)) is given by:

$$\begin{aligned}
\phi^n(H^n) &= \int_{[-\infty,\infty)^2} \max(\eta_0^n, \log H^n + \eta^n) d\nu^n(\eta_0^n, \eta^n), \\
&= \int_{\mathbb{R}} \eta_0^n d\nu^n(\eta_0^n, -\infty) + \int_{\mathbb{R}} \eta^n d\nu^n(-\infty, \eta^n) + \nu^n(\{-\infty\} \times \mathbb{R}) \log H^n \\
&\quad + \int_{\mathbb{R}^2} \max(\eta_0^n, \log H^n + \eta^n) d\nu^n(\eta_0^n, \eta^n).
\end{aligned}$$

We now argue that ϕ^n is $\mathcal{C}^1$. To do so, we show that $H^n \in \mathbb{R}_{++} \mapsto \int_{\mathbb{R}^2} \max(\eta_0^n, \log H^n + \eta^n) d\nu^n(\eta_0^n, \eta^n)$ is $\mathcal{C}^1$. Let $\underline{H} > 0$. For every $H^n > \underline{H}$, note that the partial derivative $\frac{\partial}{\partial H^n} \max(\eta_0^n, \log H^n + \eta^n)$ exists ν^n-almost everywhere. That derivative is non-negative, and bounded above by the ν^n integrable function $(\eta_0^n, \eta^n) \mapsto 1/\underline{H}$. Hence, $H^n \mapsto \int_{\mathbb{R}^2} \max(\eta_0^n, \log H^n + \eta^n) d\nu^n(\eta_0^n, \eta^n)$ is differentiable, and

$$\begin{aligned}
\frac{\partial}{\partial H^n} \int_{\mathbb{R}^2} \max(\eta_0^n, \log H^n + \eta^n) d\nu^n(\eta_0^n, \eta^n) &= \int_{\mathbb{R}^2} \frac{\partial}{\partial H^n} \max(\eta_0^n, \log H^n + \eta^n) d\nu^n(\eta_0^n, \eta^n), \\
&= \int_{(\eta_0^n, \eta^n):\ \log H^n + \eta^n > \eta_0^n} \frac{1}{H^n} d\nu^n(\eta_0^n, \eta^n), \\
&= \nu^n \left(\{(\eta_0^n, \eta^n) \in \mathbb{R}^2 :\ \log H^n + \eta^n > \eta_0^n\} \right) \frac{1}{H^n}.
\end{aligned}$$

By Assumption ii-(c), this derivative is continuous in H^n. It follows that ϕ^n is $\mathcal{C}^1$, and that

$$\begin{aligned}
H^n \phi^{n\prime}(H^n) &= \nu^n(\{-\infty\} \times \mathbb{R}) + \nu^n \left(\{(\eta_0^n, \eta^n) \in \mathbb{R}^2 :\ \log H^n + \eta^n > \eta_0^n\} \right), \\
&= \nu^n \left(\{(\eta_0^n, \eta^n) \in [-\infty, \infty)^2 :\ \log H^n + \eta^n > \eta_0^n\} \right),
\end{aligned}$$

which is the probability that the consumer turns down the outside option in stage 3. Since ν^n is a probability measure, $H^n \phi^{n\prime}(H^n)$ is non-negative, non-decreasing in H^n, and bounded above by 1.

Put $\Phi^n(H^n) = \exp \phi^n(H^n)$ for every $H^n > 0$. Then, Φ^n is $\mathcal{C}^1$ and strictly positive, and $H^n \mapsto \frac{H^n \Phi^{n\prime}(H^n)}{\Phi^n(H^n)}$ is non-negative, non-decreasing, and bounded above by 1.

We can now move back to the second stage of the choice process. The expected utility derived from choosing nest n is $\phi^n(H^n) + \varepsilon^n$. Hence, the consumer chooses nest n with probability $\Phi^n(H^n) / \sum_{m \in \mathcal{M}} \Phi^m(H^m)$. The expected utility derived from turning down the outside option in stage 1 is therefore equal to $\log \sum_{m \in \mathcal{M}} \Phi^m(H^m) + \eta \equiv \log \Phi + \eta$. Hence, a consumer with type (η^0, η) turns down the outside option in stage 1 if and only if $\log \Phi + \eta \geq \eta_0$. (Again, due to Assumption ii-(b), ties are irrelevant.)

Let $\Psi(\Phi)$ be overall consumer surplus. By Assumption ii-(b), Ψ is well defined and given by

$$\begin{aligned}\Psi(\Phi) &= \int_{[-\infty,\infty)^2} \max\left(\eta_0, \log\Phi + \eta\right) d\mu(\eta_0,\eta), \\ &= \int_{\mathbb{R}} \eta_0 d\mu(\eta_0,-\infty) + \int_{\mathbb{R}} (\log\Phi + \eta) d\mu(-\infty,\eta) + \int_{\mathbb{R}^2} \max\left(\eta_0, \log\Phi + \eta\right) d\mu(\eta_0,\eta).\end{aligned}$$

We start by simplifying the term $\int_{\mathbb{R}}(\log\Phi+\eta)d\mu(-\infty,\eta)$. By Assumption ii-(b), $\eta \in \mathbb{R} \mapsto \log\Phi' + \eta$ is $\mu(-\infty,\cdot)$-integrable for every $\Phi' > 0$. This implies in particular that $\eta \mapsto \eta$ is $\mu(-\infty,\cdot)$-integrable. Let $\Phi' \neq 1$. Then, $\eta \mapsto \log\Phi' = (\log\Phi' + \eta) - \eta$ is the sum of two $\mu(-\infty,\cdot)$-integrable functions. That function is therefore $\mu(-\infty,\cdot)$-integrable as well. It follows that $\int_{\mathbb{R}} |\log\Phi'| d\mu(-\infty,\eta) < \infty$. Hence, $\mu(\{-\infty\}\times\mathbb{R}) < \infty$. This allows us to rewrite $\Psi(\Phi)$ as follows:

$$\begin{aligned}\Psi(\Phi) = &\int_{\mathbb{R}} \eta_0 d\mu(\eta_0,-\infty) + \int_{\mathbb{R}} \eta d\mu(-\infty,\eta) + \mu(\{-\infty\}\times\mathbb{R})\log\Phi \\ &+ \int_{\mathbb{R}^2} \max\left(\eta_0, \log\Phi + \eta\right) d\mu(\eta_0,\eta).\end{aligned}$$

We now argue that Ψ is $\mathcal{C}^1$. To this end, we show that the contribution to consumer surplus of consumers with finite types, given by $\tilde{\Psi}(\Phi) \equiv \int_{\mathbb{R}^2} \max\left(\eta_0, \log\Phi + \eta\right) d\mu(\eta_0,\eta)$, is $\mathcal{C}^1$. We would like to differentiate $\tilde{\Psi}$ under the integral sign. To do so, we first need to prove that $\mu(S^\Phi) < \infty$ for every $\Phi > 0$, where

$$S^\Phi = \left\{(\eta_0,\eta) \in \mathbb{R}^2 : \ \eta + \log\Phi \geq \eta_0\right\}.$$

Let $\Phi > 0$ and $\Phi' > \Phi$. Clearly, $S^\Phi \subset S^{\Phi'}$. For every $\Phi'' > 0$, define the following function:

$$g^{\Phi''} : (\eta_0,\eta) \in \mathbb{R}^2 \mapsto \begin{cases} \eta + \log\Phi'' & \text{if } (\eta_0,\eta) \in S^{\Phi''}, \\ 0 & \text{otherwise.}\end{cases}$$

Since $(\eta_0,\eta) \mapsto \max(\eta_0, \log\Phi + \eta)$ and $(\eta_0,\eta) \mapsto \max(\eta_0, \log\Phi' + \eta)$ are both μ-integrable, g^Φ and $g^{\Phi'}$ are μ-integrable as well. As a result, $g^{\Phi'} - g^\Phi$ is μ-integrable, and

$$\log(\Phi'/\Phi)\mu(S^\Phi) = \int_{S^\Phi} |\log(\Phi'/\Phi)| d\mu = \int_{S^\Phi} |g^{\Phi'} - g^\Phi| d\mu < \infty.$$

In words: For every $\Phi > 0$, the mass of consumers who turn down the outside option is finite.

We are now in a position to prove differentiability. Let $0 < \underline{\Phi} < \overline{\Phi}$. For every $\Phi \in (\underline{\Phi}, \overline{\Phi})$, the partial derivative $\frac{\partial}{\partial\Phi}\max(\eta_0, \log\Phi + \eta)$ exists for μ-almost every (η_0,η) (using Assumption ii-(b)). Moreover, that partial derivative is non-negative, and bounded above by the function

$$(\eta_0, \eta) \in \mathbb{R}^2 \mapsto \begin{cases} \frac{1}{\underline{\Phi}} & \text{if } (\eta_0, \eta) \in S^{\overline{\Phi}}, \\ 0 & \text{otherwise,} \end{cases}$$

which is μ-integrable, since $\mu(S^{\overline{\Phi}}) < \infty$. It follows that $\tilde{\Psi}$ is differentiable on $(\underline{\Phi}, \overline{\Phi})$, and

$$\tilde{\Psi}'(\Phi) = \frac{1}{\Phi}\mu(S^{\Phi}).$$

Moreover, $\tilde{\Psi}'$ is continuous (by Assumption ii-(b)) and non-negative, and $\Phi \mapsto \Phi\tilde{\Psi}'(\Phi)$ is non-decreasing (since $S^{\Phi} \subseteq S^{\Phi'}$ whenever $\Phi \leq \Phi'$). We can conclude that Ψ is $\mathcal{C}^1$, and $\Phi \mapsto \Phi\Psi'(\Phi)$ is non-negative and non-decreasing. Moreover, $\Phi\Psi'(\Phi)$ is equal to $\mu(\{(\eta_0, \eta) \in [-\infty, \infty) : \log \eta + \Phi \geq \eta^0\})$, the mass of consumers who turn down the outside option in stage 1.

To sum up, overall consumer surplus is equal to $\Psi(\Phi)$. A mass $\Phi\Psi'(\Phi)$ of consumers turn down the outside option in stage 1. Out of those consumers, a fraction Φ^n/Φ choose nest n in stage 2. Out of those consumers, a fraction $H^n\Phi^{n\prime}(H^n)/\Phi^n(H^n)$ turn down the nest-specific outside option in stage 3. Out of those consumers, a fraction $h_i(p_i)/H^n$ choose product $i \in n$ (and consume $-h_i'(p_i)/h_i(p_i)$) in stage 4. Hence, the total demand for good i is given by:

$$D_i(p) = \Phi\Psi'(\Phi) \times \frac{\Phi^n}{\Phi} \times \frac{H^n\Phi^{n\prime}(H^n)}{\Phi^n} \times \frac{h_i}{H^n} \times \frac{-h_i'}{h_i} = -h_i'\Phi^{n\prime}\Psi',$$

which is the expression given in part (ii).

(ii) $\Rightarrow$ (i). Conversely, suppose that the demand system D can be written as in part (ii) of the proposition. We need to construct a measure μ over $[-\infty, \infty)^2$ and a collection of probability measures $(\nu^m)_{m\in\mathcal{M}}$ over $[-\infty, \infty)^2$ that satisfy parts (b) and (c) of Assumption ii, and such that the multi-stage discrete/continuous choice model $(\mathcal{N}, \mathcal{M}, \mu, (\nu^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}})$ gives rise to D.

We first construct the probability measures $(\phi^m)_{m\in\mathcal{M}}$. Let $n \in \mathcal{N}$. Put $\phi^n = \log \Phi^n$. Then, ϕ^n is $\mathcal{C}^1$, and $H^n \mapsto H^n\phi^{n\prime}(H^n)$ is non-negative, non-decreasing, and bounded above by 1. We now drop the nest superscript to ease notation. Our goal is to construct a joint probability measure ν over $[-\infty, \infty)^2$ that satisfies Assumption ii-(c), and such that, for every $H > 0$,

$$\phi(H) = \int_{[-\infty,\infty)^2} \max(\eta_0, \log H + \eta) d\nu(\eta_0, \eta).$$

Clearly, $0 \leq \lim_{H\to 0} H\phi'(H) \leq \lim_{H\to\infty} H\phi'(H) \leq 1$. Put $\alpha = \lim_{H\to 0} H\phi'(H)$ and $\beta = \lim_{H\to\infty} H\phi'(H) - \alpha$. If $\beta = 0$, then $H\phi'(H)$ is constant. It follows that

$$\phi(H) = \alpha \log H + \phi(1).$$

This ϕ function can be trivially generated, e.g., by the discrete probability measure which puts weight α on the event $(\eta_0, \eta) = (-\infty, \phi(1))$, and weight $1 - \alpha$ on the event $(\eta_0, \eta) = (\phi(1), -\infty)$.

We now turn our attention to the more interesting case in which $\beta > 0$. For every $H > 0$, put

$$\tilde{\phi}(H) = \frac{1}{\beta}\left(\phi(H) - \alpha \log H\right).$$

Note that $\tilde{\phi}$ and $H\tilde{\phi}'(H)$ are non-decreasing, and that $\lim_{H\to 0} H\tilde{\phi}'(H) = 0$ and $\lim_{H\to\infty} H\tilde{\phi}'(H) = 1$.

Let Δ be a random variable with continuous cumulative distribution function $F(\delta) = 1 - \exp(-\delta)\tilde{\phi}'(\exp(-\delta))$. (It follows from the properties of $H\tilde{\phi}'(H)$ that F is indeed a cumulative distribution function.) We use Δ to define the random variables E_0 and E as follows:

$$\begin{aligned} E_0 &= \tilde{\phi}(1) - \max(0, \Delta), \\ E &= E_0 + \Delta = \tilde{\phi}(1) + \Delta - \max(0, \Delta). \end{aligned}$$

Clearly, the random variable $E - E_0 = \Delta$ is continuously distributed. Let $\tilde{\nu}$ be the joint probability distribution of (E_0, E). We need to show that

$$\int_{\mathbb{R}^2} |\max(\eta_0, \log H + \eta)| d\tilde{\nu} < \infty, \quad \forall H > 0,$$

or, equivalently,

$$\int_{\mathbb{R}^2} |\eta_0 + \max(0, \log H + \eta - \eta_0)| d\tilde{\nu} < \infty, \quad \forall H > 0.$$

By definition of the random vector (E_0, E), this is equivalent to showing that

$$I(H) = \int_{\mathbb{R}} \left|\tilde{\phi}(1) - \max(0, \delta) + \max(0, \log H + \delta)\right| dF(\delta) < \infty, \quad \forall H > 0.$$

We now simplify $I(H)$. Suppose first that $H \le 1$. Then,

$$\begin{aligned} I(H) = &\int_{-\infty}^{0} |\tilde{\phi}(1)| dF(\delta) + \int_{0}^{-\log H} |\tilde{\phi}(1) - \delta| dF(\delta) + \int_{-\log H}^{\infty} |\tilde{\phi}(1) + \log H| dF(\delta), \\ &\le |\tilde{\phi}(1)| + |\log H| < \infty. \end{aligned}$$

Similarly, if $H > 1$, then,

$$\begin{aligned} I(H) = &\int_{-\infty}^{-\log H} |\tilde{\phi}(1)| dF(\delta) + \int_{-\log H}^{0} |\tilde{\phi}(1) + \delta + \log H| dF(\delta) + \int_{0}^{\infty} |\tilde{\phi}(1) + \log H| dF(\delta), \\ &\le |\tilde{\phi}(1)| + 2|\log H| < \infty. \end{aligned}$$

Hence, $(\eta_0, \eta) \mapsto \max(\eta_0, \log H + \eta)$ is μ-integrable for every H.

For every $H > 0$, let

$$\zeta(H) = \int_{\mathbb{R}^2} \max(\eta_0, \log H + \eta) d\tilde{\nu}(\eta_0, \eta).$$

$\zeta(H)$ is the overall consumer surplus generated by the choice process ν when the inside option is worth $\log H$. Note that, by definition of (η_0, η),

$$\zeta(1) = \tilde{\phi}(1) + \int_{\mathbb{R}} (-\max(\delta, 0) + \max(0, \delta))\, dF(\delta) = \tilde{\phi}(1) = \Phi(1).$$

Moreover, since $E - E_0$ is continuously distributed, we know that ζ is differentiable (see the first part of the proof), and

$$\begin{aligned}
\zeta'(H) &= \frac{1}{H}\tilde{\nu}\left(\{(\eta_0, \eta) :\ \eta + \log H \geq \eta_0\}\right), \\
&= \frac{1}{H}\left(1 - F(-\log H)\right), \\
&= \tilde{\phi}'(H).
\end{aligned}$$

Hence, the function ζ is such that $\zeta(1) = \tilde{\phi}(1)$, and $\zeta'(H) = \tilde{\phi}'(H)$ for every $H > 0$. It follows that $\zeta = \tilde{\phi}$.

We can therefore generate the function ϕ with the probability measure ν, which is defined as follows: ν puts weight α on $\{(-\infty, 0)\}$ (and no weight on $\{-\infty\} \times ([-\infty, \infty) \setminus \{0\})$); ν puts weight $1-\alpha-\beta$ on $(0, -\infty)$ (and no weight on $([-\infty, \infty) \setminus \{0\}) \times \{-\infty\}$); the remaining weight is put on $\mathbb{R}^2$; The probability measure conditional on being in $\mathbb{R}^2$ is given by $\tilde{\nu}$. This does give rise to ϕ, since the expected utility derived from this choice process is

$$\alpha \log H + \beta\tilde{\phi}(H) + 0 = \phi(H).$$

We now construct a measure μ that gives rise to Ψ. Let $\alpha = \lim_{\Phi\to 0} \Phi\Psi'(\Phi)$. Define

$$\tilde{\Psi}(\Phi) = \Psi(\Phi) - \alpha \log \Phi - \Psi(1), \quad \forall \Phi > 0.$$

Note that $\tilde{\Psi}(1) = 0$. Moreover, $G(\Phi) \equiv \Phi\tilde{\Psi}'(\Phi)$ is continuous, non-decreasing, and goes to 0 as Φ goes to zero. Hence, G is the cumulative distribution function of a σ-finite measure ρ over $\mathbb{R}_{++}$.

Let $\gamma : x \in \mathbb{R}_{++} \mapsto -\log x \in \mathbb{R}$. Let $\lambda \equiv \gamma_*(\rho)$ be the push-forward measure of ρ, i.e., $\lambda(B) = \rho(\gamma^{-1}(B))$ for every Borel set B. Note that, for every $\delta \in \mathbb{R}$,

$$\lambda([\delta, \infty)) = \rho\left(\gamma^{-1}([\delta, \infty))\right) = \rho\left((0, \mathrm{e}^{-\delta}]\right) = G(\mathrm{e}^{-\delta}) < \infty.$$

It follows that λ is σ-finite. Moreover, by continuity of G, we also have that $\lambda(\{\delta\}) = 0$ for every $\delta \in \mathbb{R}$.

We now use λ to construct μ, a measure over $\mathbb{R}^2$. Let

$$\chi : \delta \in \mathbb{R} \mapsto (-\max(0,\delta), \delta - \max(0,\delta)) \in \mathbb{R}^2.$$

χ is continuous, hence, measurable. Let μ be the push-forward measure of λ: $\mu \equiv \chi_*(\lambda)$. Note that, for every $X \in \mathbb{R}$,

$$\mu\left(\{(\eta_0,\eta) \in \mathbb{R}^2 :\ \eta + X = \eta_0\}\right) = \lambda\left(\{-X\}\right) = 0,$$

so the atomless part of Assumption ii-(b) holds for the measure μ.

We also argue that μ is σ-finite. To see this, consider the following sequence of sets: $B^n = (-n,\infty)^2$ for every $n \geq 1$. Clearly, $\bigcup_{n\geq 1} B^n = \mathbb{R}^2$. Moreover, for every $n \geq 1$,

$$\begin{aligned}
\chi^{-1}(B^n) &= \left\{\delta \in \mathbb{R} :\ (-\max(0,\delta), \delta - \max(0,\delta)) \in (-n,\infty)^2\right\}, \\
&= \left\{\delta \in \mathbb{R} :\ -\max(0,\delta) > -n \text{ and } \delta - \max(0,\delta) > -n\right\}, \\
&= \left\{\delta \in \mathbb{R}_+ :\ -\delta > -n \text{ and } 0 > -n\right\} \cup \left\{\delta \in \mathbb{R}_- :\ 0 > -n \text{ and } \delta > -n\right\}, \\
&= [0,n) \cup (-n,0] = (-n,n) \subset [-n,\infty).
\end{aligned}$$

Hence, $\mu(B^n) \leq \lambda([-n,\infty)) < \infty$, and μ is σ-finite.

We can now use the change-of-variables formula to prove that $(\varepsilon_0,\varepsilon) \mapsto \max(\varepsilon_0, \log\Phi + \varepsilon)$ is μ-integrable for every $\Phi > 0$:

$$\begin{aligned}
\int_{\mathbb{R}^2} |\max(\varepsilon_0, \log\Phi + \varepsilon)| d\mu &= \int_{\mathbb{R}^2} |\varepsilon_0 + \max(0, \log\Phi + \varepsilon - \varepsilon_0)|\, d\mu, \\
&= \int_{\mathbb{R}} |\chi_1(\delta) + \max\left(0, \log\Phi + \chi_2(\delta) - \chi_1(\delta)\right)|\, d\lambda(\delta), \\
&= \int_{\mathbb{R}} |-\max(0,\delta) + \max\left(0, \log\Phi + \delta\right)|\, d\lambda(\delta), \\
&\equiv I(\Phi).
\end{aligned}$$

If $\Phi \geq 1$, then

$$\begin{aligned}
I(\Phi) &= \int_{-\log\Phi}^{0} |\log\Phi + \delta| d\lambda(\delta) + \int_0^\infty |\log\Phi| d\lambda(\delta), \\
&\leq 2(\log\Phi)\lambda\left([-\log\Phi,\infty)\right) < \infty.
\end{aligned}$$

If $\Phi < 1$, then

$$\begin{aligned}
I(\Phi) &= \int_0^{-\log\Phi} |\delta| d\lambda(\delta) + \int_{-\log\Phi}^{\infty} |\log\Phi| d\lambda(\delta), \\
&\leq |\log\Phi|\lambda\left([0,\infty)\right) < \infty.
\end{aligned}$$

Hence, $(\varepsilon_0, \varepsilon) \mapsto \max(\varepsilon_0, \log\Phi + \varepsilon)$ is μ-integrable for every $H > 0$. Moreover, by the change-of-variables formula,

$$\zeta(\Phi) \equiv \int_{\mathbb{R}^2} \max(\varepsilon_0, \log\Phi + \varepsilon) d\mu = \int_{\mathbb{R}} \left(-\max(0,\delta) + \max\left(0, \log\Phi + \delta\right)\right) d\lambda(\delta).$$

In particular, $\zeta(1) = 0$. Moreover, as shown in the first part of the proof, ζ is differentiable, and

$$\begin{aligned}
\zeta'(\Phi) &= \frac{1}{\Phi}\mu\left(\left\{(\varepsilon_0,\varepsilon) \in \mathbb{R}^2 : \ \varepsilon + \log\Phi \geq \varepsilon_0\right\}\right), \\
&= \frac{1}{\Phi}\lambda\left(\left\{\delta \in \mathbb{R} : \ \delta + \log\Phi \geq 0\right\}\right), \\
&= \frac{1}{\Phi}G(\Phi), \\
&= \tilde{\Psi}'(\Phi).
\end{aligned}$$

It follows that $\zeta = \tilde{\Psi}$.

We can then extend the measure μ to $[-\infty,\infty)^2$ by adding the mass points $\mu(\{(-\infty,0)\}) = \alpha$ and $\mu(\{(\Psi(1),-\infty)\}) = 1$. Clearly, the extended μ continues to satisfy Assumption ii-(b). It is then immediate that, for every $\Phi > 0$,

$$\begin{aligned}
\int_{[-\infty,\infty)^2} \max\left(\eta_0, \eta + \log\Phi\right) d\mu(\eta_0,\eta) &= \alpha\log\Phi + \Psi(1) + \zeta(\Phi), \\
&= \alpha\log\Phi + \Psi(1) + \tilde{\Psi}(\Phi), \\
&= \Psi(\Phi).
\end{aligned}$$

Hence, μ gives rise to Ψ. □

Proposition IX implies that a demand system that can be derived from multi-stage discrete/continuous choice is fully characterized by the tuple $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}})$, where the Ψ, Φ and h functions satisfy conditions (a), (b), and (c) in the statement of the proposition. This class of demand systems generalizes the one defined in Section I along two dimensions. First, the nest partition $\mathcal{M}$ and the profile of functions $(\Phi^m)_{m\in\mathcal{M}}$ allow us to obtain substitution patterns between products that go beyond those implied by the IIA property. Second, the function Ψ permits arbitrary substitution patterns between the products in $\mathcal{N}$ and the outside option. In the following, we identify the discrete/continuous choice model $(\mathcal{N}, \mathcal{M}, \mu, (\nu^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}})$ with the tuple $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}})$ it induces. Any such tuple should be understood as satisfying the conditions in the statement of Proposition IX.

Exogenously priced products. Let $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}})$ be a demand system derivable from multi-stage discrete/continuous choice. Suppose that the products in nest n^0 are exogenously priced according to $(p_j)_{j\in n^0} \in (0,\infty]^\infty$, and let $\Phi^0 = \Phi^{n^0}\left(\sum_{j\in n^0} h_j(p_j)\right) \geq 0$.

Let $\mathcal{M}' = \mathcal{M} \setminus \{n^0\}$ and $\mathcal{N}' = \mathcal{N} \setminus n^0$. Then, it is straightforward to show that the demand system

$$D_i(p) = -h_i'(p_i)\Phi^{n\prime}\left(\sum_{j\in n} h_j(p_j)\right)\Psi'\left(\Phi^0 + \sum_{m\in\mathcal{M}'}\Phi^m\left(\sum_{k\in m} h_k(p_k)\right)\right), \ \forall p \in \mathbb{R}_{++}^{\mathcal{N}'}, \ \forall i \in n \in \mathcal{M}'$$

can still be derived from multi-stage discrete/continuous choice. In the following, we denote this demand system by $(\Psi, (\Phi^m)_{m\in\mathcal{M}'}, (h_j)_{j\in\mathcal{N}'}, \Phi^0)$, and we interpret Φ^0 as the value of the outside option.

Examples. If $\Psi(\Phi) = \log(\Phi + \Phi^0)$, where $\Phi^0 \geq 0$ is a parameter, $\Phi^m(H^m) = (H^m)^\alpha$ for all $m \in \mathcal{M}$, where $\alpha \in (0,1)$ is a parameter, and $h_j(p_j) = a_j p_j^{1-\sigma}$ for all $j \in \mathcal{N}$, where $a_j > 0$ and $\sigma > 1$ are parameters, then we obtain the nested CES demand system. If $\Psi(\Phi) = \log(\Phi + \Phi^0)$ (with $\Phi^0 \geq 0$), $\Phi^m(H^m) = (H^m)^\alpha$ for all $m \in \mathcal{M}$ ($\alpha \in (0,1)$), and $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda}}$ for all $j \in \mathcal{N}$ (with $a_j \in \mathbb{R}$ and $\lambda > 1$), then we obtain the nested MNL demand system.

Heterogeneity. It is clear that this more general discrete/continuous choice process can still accommodate the kind of *ex post* consumer heterogeneity described at the end of Section I.1, as long as consumers observes their types only after having chosen a product. As already discussed in that section, if consumers observes their types before deciding which product to patronize, then the demand system becomes a mixture of equation (xxi), and, in general, the associated pricing game loses its aggregative properties.

A particular type of *ex ante* heterogeneity can however be accommodated, where the h functions take the additively separable form $h_i(p_i, t) = h_i(p_i) + t_i$, where $t \in \mathbb{R}_{++}^{\mathcal{N}}$ is the consumer's type. To see this, suppose that each consumer type t's choice process is described by the discrete/continuous choice model $((h_j(\cdot, t))_{j\in\mathcal{N}}, H^0)$. Note that consumers are heterogeneous both in terms of conditional demand $(-h_i'(p_i)/(h_i(p_i) + t_i))$, but also in terms of choice probabilities $(\frac{h_i(p_i)+t_i}{H^0 + \sum_{j\in\mathcal{N}} h_j(p_j)+t_j})$. Suppose also that t is drawn from a finite measure λ with compact support T. It follows from our analysis in Section I.1 that overall consumer surplus at price vector p is given by

$$V(p) = \int_T \log\left(H^0 + \sum_{j\in\mathcal{N}} h_j(p_j) + t_j\right) d\lambda(t) \equiv \Psi\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right),$$

whereas the total demand for product i is given by

$$D_i(p) = \int_T \frac{-h_i'(p_i)}{H^0 + \sum_{j\in\mathcal{N}} h_j(p_j) + t_j} d\lambda(t) = -h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right).$$

Hence, the demand system we obtain coincides with the one that can be derived from the multi-stage discrete/continuous choice process $(\Psi, \Phi, (h_j)_{j\in\mathcal{N}})$, where Ψ has been defined above, and Φ is the identity function. Note that, for every $\Phi > 0$,

$$\Phi\Psi'(\Phi) = \int_T \frac{\Phi}{H^0 + \Phi + t_j} d\lambda(t),$$

which is non-negative, continuous and non-decreasing in Φ. Hence, $(\Psi, \Phi, (h_j)_{j\in\mathcal{N}})$ does satisfy conditions (a), (b) and (c) in Proposition IX.

VII.2 Representative Consumer Approach

We now show that the demand system (xxi) can also be derived from the maximization of the utility function of a representative consumer with quasi-linear preferences:

Proposition X. *Let D be the demand system generated by the multi-stage discrete/continuous choice model $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}}, \Phi^0)$. D is quasi-linearly integrable. Moreover, v is an indirect subutility function for D if and only if there exists a constant $\alpha \in \mathbb{R}$ such that*

$$v(p) = \alpha + \Psi\left(\Phi^0 + \sum_{m\in\mathcal{M}} \Phi^m\left(\sum_{j\in m} h_j(p_j)\right)\right), \quad \forall p \in \mathbb{R}^{\mathcal{N}}_{++}.$$

Proof. Clearly, $V : p \in \mathbb{R}^{\mathcal{N}}_{++} \mapsto \Psi\left(\Phi^0 + \sum_{m\in\mathcal{M}} \Phi^m\left(\sum_{j\in m} h_j(p_j)\right)\right)$ is a potential for the vector field D. By Theorem 1 in Nocke and Schutz (2017), all we need to do is check that V is convex.

For every $m \in \mathcal{M}$ and $X \in \mathbb{R}$, define $\widetilde{\Psi}(X) = \Psi(\exp X)$ and $\widetilde{\Phi}^m(X) = \log(\Phi^m(\exp X))$. Note that, by conditions (b) and (c) in Proposition IX, $\widetilde{\Psi}'(X) = \mathrm{e}^X \Psi'(\mathrm{e}^X)$ and $\widetilde{\Phi}^{m\prime}(X) = \frac{\mathrm{e}^X \Phi^{m\prime}(\mathrm{e}^X)}{\Phi^m(\mathrm{e}^X)}$ are both non-negative and non-decreasing. Hence, $\widetilde{\Psi}$ and $\widetilde{\Phi}^m$ are non-decreasing and convex.

The function V can be reexpressed as follows:

$$V(p) = \widetilde{\Psi}\left(\log\left(\Phi^0 + \sum_{m\in\mathcal{M}} \exp\left(\widetilde{\Phi}^m\left(\log\sum_{j\in m} h_j(p_j)\right)\right)\right)\right).$$

Let $m \in \mathcal{M}$. For every $j \in m$, h_j is log-convex. It follows that $(p_j)_{j\in m} \mapsto \sum_{j\in m} h_j(p_j)$ is log-convex as well. Hence, $(p_j)_{j\in m} \mapsto \widetilde{\Phi}^m\left(\log\sum_{j\in m} h_j(p_j)\right)$, which is the composition of the non-decreasing and convex function $\widetilde{\Phi}^m$ and the convex function $(p_j)_{j\in m} \mapsto \log\sum_{j\in m} h_j(p_j)$, is convex. It follows that $(p_j)_{j\in m} \mapsto \exp\widetilde{\Phi}^m\left(\log\sum_{j\in m} h_j(p_j)\right)$ is log-convex, and that $p \mapsto \Phi^0 + \sum_{m\in\mathcal{M}} \exp\widetilde{\Phi}^m\left(\log\sum_{j\in m} h_j(p_j)\right)$ is log-convex as well. Hence, V, which is the composition of the convex and non-decreasing function $\widetilde{\Psi}$ and the convex function $p \mapsto \log\left(\Phi^0 + \sum_{m\in\mathcal{M}} \exp\widetilde{\Phi}^m\left(\log\sum_{j\in m} h_j(p_j)\right)\right)$ is convex. □

Just like in Section I, any demand system that can be derived from multi-stage discrete/continuous choice can also be derived from quasi-linear utility maximization. Moreover, the overall consumer surplus function generated under discrete/continuous choice and the indirect utility function of the associated representative consumer coincide (up to an additive constant).

VIII Multi-Product Firm Pricing Games and Nested Demand Systems

VIII.1 Definition of the Pricing Game

A pricing game is a tuple $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}}, \Phi^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$, where $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}}, \Phi^0)$ is a nested demand system, as studied in Section VII, $\mathcal{F}$, the set of firms, is a partition of $\mathcal{N}$ containing at least two elements, and $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$ is the marginal costs vector. Throughout this section, we maintain the assumption that the nest partition $\mathcal{M}$ is a coarsening of the firm partition f. This means that a given nest m can contain products owned by different firms, but a firm is present in only one nest. In the following, we will often abuse notation, and write $f \in m$ when firm f's set of products is contained in nest m.

The profit of firm $f \in m \in \mathcal{M}$ is defined as follows: For every $p \in (0,\infty]^{\mathcal{N}}$,

$$\Pi^f(p) = \sum_{\substack{j\in f\\ p_j<\infty}} (p_j - c_j)(-h_j'(p_j))\Phi^{m\prime}\left(\sum_{g\in m}\sum_{k\in g} h_k(p_k)\right)\Psi'\left(\Phi^0 + \sum_{n\in\mathcal{M}}\Phi^n\left(\sum_{g\in n}\sum_{k\in g} h_k(p_k)\right)\right),$$

where we continue to use the notation $h_j(\infty) = \lim_{p_j\to\infty} h_j(p_j)$.

We make the following assumptions:

Assumption iii. *(a) For every $j \in \mathcal{N}$, h_j is a $\mathcal{C}^3$, strictly decreasing, and log-convex function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$.*

(b) For every $m \in \mathcal{M}$, Φ^m is a $\mathcal{C}^2$ function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Moreover, $\varphi^m : H^m \mapsto \frac{H^m\Phi^{m\prime}(H^m)}{\Phi^m(H^m)}$ is strictly positive, non-decreasing, and bounded above by 1.

(c) Ψ is a $\mathcal{C}^2$ function from $\mathbb{R}_{++}$ to $\mathbb{R}$, and $\Phi^0 \geq 0$. Moreover, $\Phi \mapsto \Phi\Psi'(\Phi)$ is strictly positive and non-decreasing.

(d) For every $m \in \mathcal{M}$, $\Phi^{m\prime\prime} \leq 0$.

(e) $\Psi'' < 0$.

(f) For every $j \in \mathcal{N}$ and $p_j > 0$, $\iota_j'(p_j) \geq 0$ whenever $\iota_j(p_j) > 1$.

(g) For every $f \in \mathcal{F}$, at least one of the following conditions holds true:

- $\min_{j\in f}\inf_{p_j>\underline{p}_j}\rho_j(p_j) \geq \max_{j\in f}\sup_{p_j>\underline{p}_j}\theta_j(p_j)$.

– $\bar{\mu}^f \leq \mu^*(\simeq 2.78)$, *and for every* $j \in f$, $\bar{\mu}_j = \bar{\mu}^f$, $\lim_{p_j \to \infty} h_j(p_j) = 0$ *and* ρ_j *is non-decreasing on* $(\underline{p}_j, \infty)$.

– *There exist a function* $h^f \in \mathcal{H}^\iota$, *a collection of quality weights* $(a_j)_{j \in f} \in \mathbb{R}^f_{++}$, *and a marginal cost level* $c^f > 0$ *such that* $h_j = a_j h^f$ *and* $c_j = c^f$ *for all* $j \in f$. *In addition,* ρ^f *is non-decreasing on* $(\underline{p}, \infty)$.

(h) For every $m \in \mathcal{M}$, $\vartheta^m : H^m \mapsto \frac{H^m(-\Phi^{m\prime\prime}(H^m))}{\Phi^{m\prime}(H^m)}$ *is non-decreasing.*

(i) $\eta : \Phi \mapsto \frac{\Phi(-\Psi''(\Phi))}{\Psi'(\Phi)}$ *is non-decreasing.*

Assumptions iii–(a)–(c) mean that the demand system can be derived from multi-stage discrete/continuous choice, and that demand is smooth and never vanishes. Assumptions iii–(d) and (e) imply that products are substitutes. (In general, products can be complements under multi-stage discrete/continuous choice due to a one-stop shopping effect: When p_i decreases, more consumers turn down the outside option in stages 1 and 3 of the discrete/continuous choice process; This can end up boosting the demand for product j, despite the fact that consumers have incentives to substitute towards product i.) Assumption iii–(f) is the same as Assumption 1 in the paper. It ensures, among other things, that first-order conditions are sufficient for global optimality. Assumptions iii–(g)–(i) will play a similar role in the analysis. Note that Assumption iii–(g) is simply the uniqueness condition stated in Theorem II.

VIII.2 Equilibrium Existence, Uniqueness, and Characterization

Fix a pricing game $(\Psi, (\Phi^m)_{m \in \mathcal{M}}, (h_j)_{j \in \mathcal{N}}, \Phi^0, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$, where $(\Psi, (\Phi^m)_{m \in \mathcal{M}}, (h_j)_{j \in \mathcal{N}}, \Phi^0)$ satisfies Assumption iii. In this section, we show that the pricing game has a unique equilibrium. The approach is similar to the one in Section A of the paper, in that the equilibrium existence and uniqueness problem can be re-expressed as a nested fixed point problem. An important difference with the approach in the paper is that the game is no longer fully aggregative, in the sense that firm f's profit ($f \in n$) now depends not only on the prices it sets and the aggregator level $\Phi = \Phi^0 + \sum_{m \in \mathcal{M}} \Phi^m(\sum_{j \in m} h_j(p_j))$, but also on the value of the sub-aggregator $H^n = \sum_{j \in n} h_j(p_j)$.

We start by proving the following technical lemma:

Lemma XV. *(a) For every* $m \in \mathcal{M}$ *and* $j \in m$, $\lim_{p_j \to \infty} p_j h_j'(p_j) \Phi^{m\prime}(h_j(p_j)) = 0$.

(b) For every $f \in \mathcal{F}$ *such that* $\lim_{p_j \to \infty} h_j(p_j) = 0$ *for every* $j \in f$,

$$\lim_{\mu^f \to \bar{\mu}^f} \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f} h_j(r_j(\mu^f))}{\sum_{j \in f} \gamma_j(r_j(\mu^f))} \geq 1.$$

(c) For every $m \in \mathcal{M}$, $H^m \mapsto H^m \Phi^{m\prime}$ *is strictly increasing.*

(d) For every $m \in \mathcal{M}$, $\lim_{H^m \to 0} H^m \Phi^{m\prime}(H^m) = 0$.

(e) For every $m \in \mathcal{M}$, $\lim_{H^m \to 0} \vartheta^m(H^m) < 1$.

(f) For every $m \in \mathcal{M}$ such that $\lim_{H^m \to 0} \Phi^m(H^m) = 0$,

$$\lim_{H^m \to 0} \varphi^m(H^m) = 1 - \lim_{H^m \to 0} \vartheta^m(H^m).$$

(g) $\eta(\Phi) \leq 1$ for every $\Phi > 0$.

Proof. **(a)** Let $\xi_j(p_j) = \Phi^n(h_j(p_j))$. Note that, by Assumptions iii–(a) and (b), $\xi_j > 0$, $\xi_j' < 0$, and

$$\frac{d \log \xi_j}{dp_j} = -\frac{-h_j'(p_j)}{h_j(p_j)} \frac{h_j(p_j)\Phi^{n\prime}(h_j(p_j))}{\Phi^n(h_j(p_j))}$$

is non-decreasing in p_j. Hence, ξ_j is strictly positive, strictly decreasing and log-convex. By Lemma A–(a),

$$0 = \lim_{p_j \to \infty} p_j \xi_j'(p_j) = \lim_{p_j \to \infty} p_j h_j'(p_j) \Phi^{n\prime}(h_j(p_j)).$$

(b) Assume first that $\bar{\mu}^f < \infty$. Let $f' = \{j \in f : \bar{\mu}_j = \bar{\mu}^f\}$. Then, for μ^f sufficiently high,

$$\frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f} h_j(r_j(\mu^f))}{\sum_{j \in f} \gamma_j(r_j(\mu^f))} = \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f'} h_j(r_j(\mu^f))}{\sum_{j \in f'} \gamma_j(r_j(\mu^f))}.$$

(Recall that $\gamma_j(\infty) = 0$ by Lemma A, and, by assumption, $h_j(\infty) = 0$.) Let $\varepsilon > 0$. Recall that $\lim_{p_j \to \infty} \rho_j(p_j) = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}$ for every $j \in f'$ (Lemma A). Hence, there exists $\underline{\mu} < \bar{\mu}^f$ such that, for every $j \in f'$,

$$\frac{\bar{\mu}^f - 1}{\bar{\mu}^f} - \varepsilon \leq \rho_j(r_j(\mu^f)) \leq \frac{\bar{\mu}^f - 1}{\bar{\mu}^f} + \varepsilon$$

for every $\mu^f > \underline{\mu}$. Rewriting, this means that

$$\gamma_j(r_j(\mu^f)) \left(\frac{\bar{\mu}^f}{\bar{\mu}^f - 1} - \varepsilon \right) \leq h_j(r_j(\mu^f)) \leq \gamma_j(r_j(\mu^f)) \left(\frac{\bar{\mu}^f}{\bar{\mu}^f - 1} + \varepsilon \right),$$

for every $\mu^f > \underline{\mu}$. Adding up, and dividing by $\sum_{j \in f'} \gamma_j(r_j(\mu^f))$, we obtain:

$$\frac{\bar{\mu}^f}{\bar{\mu}^f - 1} - \varepsilon \leq \frac{\sum_{j \in f'} h_j(r_j(\mu^f))}{\sum_{j \in f'} \gamma_j(r_j(\mu^f))} \leq \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} + \varepsilon$$

for every $\mu^f > \underline{\mu}$. It follows that $\lim_{\mu^f \to \bar{\mu}^f} \frac{\sum_{j \in f} h_j(r_j(\mu^f))}{\sum_{j \in f} \gamma_j(r_j(\mu^f))} = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}$, which proves part (b) when $\bar{\mu}^f < \infty$.

Next, assume instead that $\bar{\mu}^f = \infty$. By Lemmas VII–IX and Assumptions iii–(f) and (g), the function $\mu^f \mapsto \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f} h_j(r_j(\mu^f))}{\sum_{j \in f} \gamma_j(r_j(\mu^f))}$ is monotone, and therefore has a limit as μ^f tends to infinity. Moreover, by log-convexity, we have that

$$\frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f} h_j(r_j(\mu^f))}{\sum_{j \in f} \gamma_j(r_j(\mu^f))} \geq \frac{\mu^f - 1}{\mu^f} \underset{\mu^f \to \infty}{\longrightarrow} 1.$$

(c) This follows immediately from the fact that, by Assumption iii–(b), $H^m \Phi^{m\prime}(H^m)/\Phi^m(H^m)$ is non-decreasing, and $\Phi(H^m)$ is strictly increasing.

(d) Let $\xi(x) = \Phi^m(\exp(-x))$ for every $x > 0$. Since $x \mapsto \mathrm{e}^{-x}$ is log-convex, part (a) implies that $\lim_{x \to \infty} \xi'(x) = 0$. Hence,

$$\lim_{H^m \to 0} H^m \Phi^{m\prime}(H^m) = \lim_{x \to \infty} \mathrm{e}^{-x} \, \Phi^{m\prime}(\mathrm{e}^{-x}) = - \lim_{x \to \infty} \xi'(x) = 0.$$

(e) Assume for a contradiction that

$$\lim_{H^m \to 0} \frac{H^m(-\Phi^{m\prime\prime}(H^m))}{\Phi^{m\prime}(H^m)} \geq 1.$$

(By Assumption iii–(h), the limit exists.) Then, by Assumption iii–(h), $\frac{H^m(-\Phi^{m\prime\prime}(H^m))}{\Phi^{m\prime}(H^m)} \geq 1$ for every $H^m > 0$. Put differently, $\frac{d}{dH^m}\left(H^m \Phi^{m\prime}(H^m)\right) \leq 0$. Since $\Phi^{m\prime} > 0$, it follows that $\frac{d}{dH^m} \frac{H^m \Phi^{m\prime}(H^m)}{\Phi^m(H^m)} < 0$, which violates Assumption iii–(b).

(f) Note that

$$\begin{aligned}
1 - \lim_{H^m \to 0} \vartheta^m(H^m) &= \lim_{H^m \to 0} \frac{\Phi^{m\prime}(H^m) + H^m \Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)}, \\
&= \lim_{H^m \to 0} \frac{\frac{d}{dH^m}\left(H^m \Phi^{m\prime}(H^m)\right)}{\frac{d}{dH^m}\left(\Phi^{m\prime}(H^m)\right)}, \\
&= \lim_{H^m \to 0} \frac{H^m \Phi^{m\prime}(H^m)}{\Phi^m(H^m)}, \\
&= \lim_{H^m \to 0} \varphi^m(H^m),
\end{aligned}$$

where the third line follows by L'Hospital's rule (by assumption, $\lim_{H^m \to 0} \Phi^m(H^m) = 0$; by part (c), $\lim_{H^m \to 0} H^m \Phi^{m\prime}(H^m) = 0$).

(g) By Assumption iii–(i), $\Phi \Psi'(\Phi)$ is non-decreasing. Therefore, $\Phi \Psi''(\Phi) + \Psi'(\Phi) \geq 0$, and $\eta(\Phi) \leq 1$. □

As in Section A, it is obvious that each firm sets at least one finite price in any equilibrium:

Lemma XVI. *In any Nash equilibrium $(p_j^*)_{j \in \mathcal{N}}$, for every firm $f \in \mathcal{F}$, there exists $k \in f$ such that $p_k^* < \infty$.*

Proof. Straightforward. □

Fix a firm $f \in n \in \mathcal{M}$. Define $H^0 = \sum_{j \in n \setminus f} h_j(p_j)$ and $\Phi^{0\prime} = \Phi^0 + \sum_{m \in \mathcal{M} \setminus \{n\}} \Phi^m(\sum_{j \in m} h_j(p_j))$. By Lemma XVI, $H^0 > 0$ or $\Phi^{0\prime} > 0$. Define also

$$G^f((p_j)_{j \in f}, H^0, \Phi^{0\prime}) = \sum_{\substack{k \in f \\ p_k < \infty}} (p_k - c_k)(-h'_k(p_k))\Phi^{n\prime}\left(\sum_{j \in f} h_j(p_j) + H^0\right) \times \Psi'\left(\Phi^n\left(\sum_{j \in f} h_j(p_j) + H^0\right) + \Phi^{0\prime}\right). \quad \text{(xxii)}$$

Note that $G^f((p_j)_{j \in f}, H^0, \Phi^{0\prime})$ is the profit of firm f when it sets price vector $(p_j)_{j \in f}$ and its rivals set price vector $(p_j)_{j \in \mathcal{N} \setminus \{f\}}$. We study the following maximization problem:

$$\max_{(p_j)_{j \in f} \in (0, \infty]^f} G^f\left((p_j)_{j \in f}, H^0, \Phi^{0\prime}\right). \quad \text{(xxiii)}$$

We now extend Lemma C:

Lemma XVII. *Maximization problem* (xxiii) *has a solution. Moreover, if* $(p_j)_{j \in f}$ *solves that maximization problem, then* $p_j \geq c_j$ *for all* $j \in f$*, and* $p_k < \infty$ *for some* $k \in f$*.*

Proof. The fact that the firm does not price below cost at any optimum follows immediately from Assumptions iii–(d) and (e). Since $G^f((\infty, \ldots, \infty), H^0, \Phi^{0\prime}) = 0$, setting only infinite prices cannot be optimal.

To show that the maximization problem has a solution, we now argue that $\lim_{p^f \to \hat{p}^f} G^f(p^f, H^0, \Phi^{0\prime}) = G^f(\hat{p}^f, H^0, \Phi^{0\prime})$ for every $\hat{p}^f \in \prod_{j \in f}[c_j, \infty]$. If the price vector $\hat{p}^f$ has at least one finite component, then this follows from Lemma A–(a) and from taking limits term by term. Suppose now that $\hat{p}^f$ only has infinite components. If $H^0 > 0$, then limits can again be taken term by term:

$$\lim_{p^f \to \hat{p}^f} G^f(p^f, H^0, \Phi^0) = \lim_{p^f \to \hat{p}^f}\left(\sum_{j \in f}(p_j - c_j)(-h'_j(p_j))\right) \times \Phi^{n\prime}\left(\sum_{k \in f} \lim_{p_k \to \infty} h_k(p_k) + H^0\right) \times \Psi'\left(\Phi^n\left(\sum_{k \in f} \lim_{p_k \to \infty} h_k(p_k) + H^0\right) + \Phi^{0\prime}\right),$$

which is indeed equal to zero by Lemma A–(a), and since $H^0 > 0$.

Assume instead that $H^0 = 0$.[12] Then, $\Phi^{0\prime} > 0$. Hence, for every $p^f \neq \hat{p}^f$,

$$G^f(p^f, 0, \Phi^{0\prime}) \leq \Psi'(\Phi^{0\prime}) \times \sum_{\substack{k \in f \\ p_k < \infty}} \underbrace{p_k(-h'_k(p_k))\Phi^{n\prime}(h_k(p_k))}_{\underset{p_k \to \infty}{\longrightarrow} 0} \underset{p^f \to \hat{p}^f}{\longrightarrow} 0,$$

[12] Note that this can happen only if firm f owns all the products in nest n.

where we have used Lemma XV–(a) and Assumptions iii–(d) and (e).

We can conclude: $G^f(\cdot, H^0, \Phi^{0\prime})$ is continuous over the compact set $\prod_{j\in f}[c_j, \infty]$. Therefore, that function has a maximum. $\square$

The generalized first-order conditions for maximization problem (xxiii) are defined as in Section A. It is obvious that they are necessary for optimality:

Lemma XVIII. *If $(p_j)_{j\in f} \in (0,\infty]^f$ solves maximization problem* (xxiii)*, then the generalized first-order conditions are satisfied at $(p_j)_{j\in f}$.*

The definition of the common ι-markup property is the same as in Section A. We now exploit that property to simplify firm f's profile of generalized first-order conditions:

Lemma XIX. *Suppose that the generalized first-order conditions for maximization problem* (10) *hold at price vector $(p_j)_{j\in f} \in (0,\infty]^f$. Then, $(p_j)_{j\in f}$ satisfies the common ι-markup property. The corresponding ι-markup, μ^f, solves the following equation on interval* $(1,\infty)$*:*

$$\frac{\mu^f - 1}{\mu^f} = \left(\sum_{j\in f} \gamma_j(r_j(\mu^f))\right)\left(\frac{-\Phi^{n\prime\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f)) + H^0\right)}{\Phi^{n\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f)) + H^0\right)} + \Phi^{n\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f)) + H^0\right)\frac{-\Psi''\left(\Phi^n\left(\sum_{j\in f} h_j(r_j(\mu^f)) + H^0\right) + \Phi^{0\prime}\right)}{\Psi'\left(\Phi^n\left(\sum_{j\in f} h_j(r_j(\mu^f)) + H^0\right) + \Phi^{0\prime}\right)}\right). \tag{xxiv}$$

In addition, the value of the objective function at this profile of prices is equal to

$$(\mu^f - 1)\frac{\left(\Phi^{n\prime}\Psi'\right)^2}{\left(\Phi^{n\prime}\right)^2(-\Psi'') + \left(-\Phi^{n\prime\prime}\right)\Psi'}.$$

Proof. Suppose the generalized first-order conditions hold at price vector $(p_j)_{j\in f}$. Assume without loss of generality that $f = \{1,\dots,n\}$, and that there exists $1 \le K \le n$ such that $p_k < \infty$ for every $1 \le k \le K$, and $p_k = \infty$ for every $K+1 \le k \le n$. For every $k \in \{1,\dots,K\}$, the derivative of firm f's profit with respect to p_k, evaluated at $(p_j)_{j\in f}$, is given by:

$$\begin{aligned}\frac{\partial G^f}{\partial p_k} &= \Psi'\Phi^{n\prime}\left(-h_k' - (p_k - c_k)h_k''\right) + \left(\sum_{j=1}^{K}(p_j - c_j)(-h_j')\right)h_k'\left(\Psi''(\Phi^{n\prime})^2 + \Psi'\Phi^{n\prime\prime}\right),\\ &= (-h_k')\Psi'\Phi^{n\prime}\left(1 - \nu_k - G^f\frac{\Psi''(\Phi^{n\prime})^2 + \Psi'\Phi^{n\prime\prime}}{\left(\Psi'\Phi^{n\prime}\right)^2}\right),\end{aligned}$$

which must be equal to zero, since the generalized first-order conditions hold at $(p_j)_{j\in f}$. Hence, there exists $\mu^f \in (1, \max_{1\le i\le K}\bar{\mu}_i)$ such that $\nu_k(p_k) = \mu^f$ (or, equivalently, $p_k =$

$r_k(\mu^f))$ for every $k \in \{1, \ldots, K\}$. This μ^f is pinned down by

$$\mu^f = 1 - \frac{\Psi''(\Phi^{f\prime})^2 + \Psi'\Phi^{f\prime\prime}}{(\Psi'\Phi^{f\prime})^2} G^f\left((p_j)_{j\in f}, H^0, \Phi^{0\prime}\right), \tag{xxv}$$

where Ψ and its derivatives are evaluated at $\Phi^{0\prime} + \Phi^n(H^0 + \sum_{j\in f} h_j(p_j))$, and Φ^n and its derivatives are evaluated at $H^0 + \sum_{j\in f} h_j(p_j)$.

Assume for a contradiction that $\mu^f < \bar{\mu}_i$ for some $i \in \{K+1, \ldots, N\}$. Let $\tilde{G}^f(x)$ be the profit of firm f when it prices product i at x, other products are priced according to $(p_j)_{j\in f}$, and other firms' prices give rise to $\Phi^{0\prime}$ and H^0. We have already shown that $\lim_{x\to\infty} \tilde{G}^f(x) = G^f((p_j)_{j\in f}, H^0, \Phi^{0\prime})$ (see the proof of Lemma XVII). Moreover,

$$\tilde{G}^{f\prime}(x) = (-h_i')\Psi'\Phi^{n\prime}\left(1 - \nu_i(x) - \tilde{G}^f(x)\frac{\Psi''(\Phi^{n\prime})^2 + \Psi'\Phi^{n\prime\prime}}{(\Psi'\Phi^{n\prime})^2}\right), \tag{xxvi}$$

where Ψ and its derivatives are evaluated at $\Phi^{0\prime} + \Phi^n(H^0 + h_i(x) + \sum_{j\in f\setminus\{i\}} h_j(p_j))$, and Φ^n and its derivatives are evaluated at $H^0 + h_i(x) + \sum_{j\in f\setminus\{i\}} h_j(p_j)$. We know from condition (xxv) that, as x tends to ∞, the term in parentheses in equation (xxvi) goes to

$$(1 - \bar{\mu}_i) - (1 - \mu^f) = \mu^f - \bar{\mu}_i < 0.$$

It follows that $\tilde{G}^f$ is strictly decreasing for x high enough. Hence, there exists $x < \infty$ such that $\tilde{G}^f(x) > \tilde{G}^f(\infty)$. It follows that the generalized first-order conditions do not hold at $(p_j)_{j\in f}$, a contradiction.

Hence, if the generalized first-order conditions hold at $(p_j)_{j\in f}$, then there exists $\mu^f \in (1, \bar{\mu}^f)$ such that $p_j = r_j(\mu^f)$ for every $j \in f$, and

$$\begin{aligned}
\mu^f &= 1 - G^f \frac{\Psi''(\Phi^{f\prime})^2 + \Psi'\Phi^{f\prime\prime}}{(\Psi'\Phi^{f\prime})^2}, \\
&= 1 - \sum_{\substack{j\in f \\ \bar{\mu}_j < \bar{\mu}^f}} (p_j - c_j)(-h_j') \frac{\Psi''(\Phi^{f\prime})^2 + \Psi'\Phi^{f\prime\prime}}{\Psi'\Phi^{f\prime}}, \\
&= 1 - \mu^f \left(\sum_{j\in f} \gamma_j\right)\left(\Phi^{f\prime}\frac{\Psi''}{\Psi'} + \frac{\Phi^{f\prime\prime}}{\Phi^{f\prime}}\right),
\end{aligned}$$

This is equivalent to equation (xxiv). The result on the value of the objective function follows immediately from equation (xxv). □

We now prove the analogue of Lemma G:

Lemma XX. *Equation* (xxiv) *has a unique solution on the interval* $(1, \infty)$.

Proof. To see why equation (xxiv) has a solution, recall that maximization problem (xxiii) has a solution p^* by Lemma XVII, that p^* satisfies the common ι-markup property by

Lemma XVIII, and that the corresponding ι-markup necessarily solves equation (xxiv) by Lemma XIX.

To prove uniqueness, note that equation (xxiv) can be rewritten as follows:

$$\underbrace{\frac{\mu^f - 1}{\mu^f}\frac{\sum_{j\in f} h_j}{\sum_{j\in f}\gamma_j}}_{A} = \frac{H^f}{H^n}\frac{H^n\Phi^{n\prime}(H^n)}{\Phi^n(H^n)}\frac{\Phi^n}{\Phi}\frac{\Phi(-\Psi''(\Phi))}{\Psi'(\Phi)} + \frac{H^f}{H^n}\frac{H^n(-\Phi^{n\prime\prime}(H^n))}{\Phi^{n\prime}(H^n)},$$

$$= \underbrace{\frac{H^f}{H^n}}_{B}\left(\underbrace{\varphi^n(H^n)}_{C}\underbrace{\frac{\Phi^n}{\Phi}}_{D}\underbrace{\eta(\Phi)}_{E} + \underbrace{\vartheta^n(H^n)}_{F}\right), \tag{xxvii}$$

where the h_j and γ_j functions are evaluated at $r_j(\mu^f)$, $H^n = H^f + H^0$ is the nest-level sub-aggregator, $H^f = \sum_{j\in f} h_j$ if firm f's contribution to that sub-aggregator, $\Phi = \Phi^{0\prime} + \Phi^n$ is the aggregator, and Φ^n is evaluated at H^n. We claim that the left-hand side of equation (xxvii) is strictly increasing in μ^f, whereas the right-hand side is strictly decreasing in μ^f. To see this, note that:

- Term A is strictly increasing in μ^f, by Lemmas VII–IX and Assumptions iii–(f) and (g).
- Term B is non-increasing in μ^f, since that term is weakly increasing in H^f (and strictly so if $H^0 > 0$), and $H^f = \sum_{j\in f} h_j(r_j(\mu^f))$ is strictly decreasing, by Assumptions iii–(a) and (f) and Lemma E.
- Term C is non-increasing in μ^f, since φ^n is non-decreasing (Assumption iii–(h)), and, as mentioned above, H^f is strictly decreasing in μ^f.
- Term D is non-increasing in μ^f, since that term is weakly increasing in Φ^n (and strictly so if $\Phi^{0\prime} > 0$), which, by Assumption iii–(b), is non-decreasing in $H^n = H^f + H^0$, which is strictly decreasing in μ^f.
- Term E is non-increasing in μ^f, since that term is non-decreasing in Φ by Assumption iii–(i), and $\Phi = \Phi^n + \Phi^{0\prime}$ is strictly decreasing in μ^f.
- Term F is non-increasing in μ^f, since that term is non-decreasing in H^n by Assumption iii–(h), and H^n is strictly decreasing in μ^f.
- (Since terms B, C, D, and E are all strictly positive, and terms B and/or D are strictly decreasing, we do obtain that the right-hand side is strictly decreasing.)

Hence, equation (xxvii) has a unique solution. □

This concludes our study of maximization problem (xxiii):

Lemma XXI. *Maximization problem* (xxiii) *has a unique solution. The generalized first-order conditions associated with this maximization problem are necessary and sufficient for global optimality. The optimal price vector (which contains at least one finite component) satisfies the common ι-markup property, and the corresponding ι-markup, μ^{f*}, is the unique solution of equation* (xxiv)*. The maximized value of the objective function is*

$$(\mu^{f*}-1)\frac{(\Phi^{n\prime}\Psi^{\prime})^2}{(\Phi^{n\prime})^2(-\Psi^{\prime\prime})+(-\Phi^{n\prime\prime})\Psi^{\prime}},$$

where Ψ and its derivatives are evaluated at $\Phi^{0\prime}+\Phi^n(H^0+\sum_{j\in f}h_j(r_j(\mu^{f*})))$, and Φ^n and its derivatives are evaluated at $H^0+\sum_{j\in f}h_j(r_j(\mu^{f*})$.

Proof. This follows immediately from Lemmas XVII–XX. □

We now turn our attention to the equilibrium existence problem. The price vector p is a Nash equilibrium if and only if, for every $n\in\mathcal{M}$ and $f\in n$, $(p_j)_{j\in f}$ maximizes

$$G^f\left(\cdot,\sum_{k\in n\setminus\{f\}}h_k(p_k),\ \Phi^0+\sum_{m\in\mathcal{M}\setminus\{n\}}\Phi^m\left(\sum_{f\in m}\sum_{k\in f}h_k(p_k)\right)\right).$$

By Lemma XXI, this is equivalent to the existence of a profile of ι-markups $(\mu^f)_{f\in\mathcal{F}}$ such that for every $n\in\mathcal{M}$ and $f\in n$,

$$\begin{aligned}\frac{\mu^f-1}{\mu^f}\frac{1}{\sum_{j\in f}\gamma_j(r_j(\mu^f))}&=\frac{-\Phi^{n\prime\prime}\left(\sum_{g\in n}\sum_{j\in g}h_j(r_j(\mu^g))\right)}{\Phi^{n\prime}\left(\sum_{g\in n}\sum_{j\in g}h_j(r_j(\mu^g))\right)}\\&+\Phi^{n\prime}\left(\sum_{g\in n}\sum_{j\in g}h_j(r_j(\mu^g))\right)\frac{-\Psi^{\prime\prime}\left(\Phi^0+\sum_{m\in\mathcal{M}}\Phi^m\left(\sum_{g\in m}\sum_{j\in g}h_j(r_j(\mu^g))\right)\right)}{\Psi^{\prime}\left(\Phi^0+\sum_{m\in\mathcal{M}}\Phi^m\left(\sum_{g\in m}\sum_{j\in g}h_j(r_j(\mu^g))\right)\right)}.\end{aligned}$$

This is, in turn, equivalent to the existence of an aggregator level Φ, a curvature level Q, a profile of sub-aggregator levels $(H^m)_{m\in\mathcal{M}}$, and a profile of ι-markups $(\mu^f)_{f\in\mathcal{F}}$ such that

$$\Phi=\Phi^0+\sum_{m\in\mathcal{M}}\Phi^m(H^m),$$

$$Q=\frac{-\Psi^{\prime\prime}(\Phi)}{\Psi^{\prime}(\Phi)},$$

$$H^m=\sum_{f\in m}\sum_{j\in f}h_j(r_j(\mu^f)),\quad\forall m\in\mathcal{M},\tag{xxviii}$$

$$\frac{\mu^f-1}{\mu^f}\frac{1}{\sum_{j\in f}\gamma_j(r_j(\mu^f))}=\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)}+\Phi^{m\prime}(H^m)Q,\quad\forall m\in\mathcal{M},\ \text{and}\ f\in m.\tag{xxix}$$

We adopt a nested fixed-point approach to solve this problem. We first show that, for every $Q > 0$ and $m \in \mathcal{M}$, there exists a unique pair $((m^f(Q))_{f\in m}, H^m(Q))$ which jointly solves equations (xxviii) and (xxix). We then show that the aggregate fitting-in function $\Phi \mapsto \Phi^0 + \sum_{m\in\mathcal{M}} \Phi^m \left(H^m \left(\frac{-\Psi''(\Phi)}{\Psi'(\Phi)}\right)\right)$ has a unique fixed point.

Lemma XXII. *For every $m \in \mathcal{M}$ and $f \in m$, for every $X \geq 0$, equation*

$$\frac{\mu^f - 1}{\mu^f} \frac{1}{\sum_{j\in f} \gamma_j(r_j(\mu^f))} = X, \tag{xxx}$$

has a unique solution in μ^f, denoted $\widetilde{m}^f(X)$. $\widetilde{m}^f$ is continuous and strictly increasing in X.

Proof. Since the left-hand side of equation (xxx) is continuous and strictly increasing in μ^f, tends to 0 as μ^f tends to 1, and tends to ∞ as μ^f tends to $\bar{\mu}^f$ (see Lemma A), whereas the right-hand side is non-negative, this equation has a unique solution. The continuity and monotonicity of $\widetilde{m}^f$ can then be established by using the same argument as in the proof of Lemma I. □

We can now define $m^f(Q)$ and $H^m(Q)$:

Lemma XXIII. *For every $m \in \mathcal{M}$, the equation*

$$H^m = \sum_{f\in m} \sum_{j\in f} h_j \left(r_j \left(\widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m) Q \right)\right)\right) \tag{xxxi}$$

has a unique solution, denoted $H^m(Q)$. $H^m(Q)$ and $m^f(Q) \equiv \widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m) Q \right)$ are continuous. Moreover, $H^m(\cdot)$ is strictly decreasing, and $m^f(\cdot)$ is strictly increasing.

Proof. Define the sub-aggregate share function

$$\Omega^m(Q, H^m) = \frac{1}{H^m} \sum_{f\in m} \sum_{j\in f} h_j \left(r_j \left(\widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m) Q \right)\right)\right).$$

Our goal is to show that the equation $\Omega^m(Q, H^m) = 1$ has a unique solution in H^m. We first show that a solution exists. By Lemma XXII, Ω^m is continuous.

We first study the behavior of Ω^m when H^m is in the neighborhood of infinity. By Lemma XV, $H^m \Phi'(H^m)$ is non-decreasing. Hence, $H^m \Phi^{m\prime\prime}(H^m) + \Phi^{m\prime}(H^m) \geq 0$. Therefore,

$$\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} \leq \frac{1}{H^m} \underset{H^m\to\infty}{\longrightarrow} 0.$$

Moreover, by Assumption iii–(b) and (d), $\Phi^{m\prime}$ is non-increasing and strictly positive. Therefore, $l = \lim_{H^m\to\infty} \Phi^{m\prime}(H^m)$ exists, and is finite and non-negative. By continuity of $\widetilde{m}^f$, it follows that

$$\lim_{H^m\to\infty} \widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m) Q \right) = \widetilde{m}^f(lQ) < \bar{\mu}^f.$$

Hence,

$$\Omega^m(Q, H^m) \underset{H^m \to \infty}{\longrightarrow} 0 \times \sum_{f \in m} \sum_{j \in f} h_j \left(r_j \left(\widetilde{m}^f \left(lQ \right) \right) \right) = 0.$$

We now study the behavior of Ω^m when H^m is in the neighborhood of zero. Assume first that, for some firm $f \in m$, $\widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m)Q \right)$ does not tend to $\bar{\mu}^f$ as H^m tends to zero. Then, there exist a sequence $(H_n^m)_{n \geq 0}$ and a ι-markup $\mu < \bar{\mu}^f$ such that $H_n^m \underset{n \to \infty}{\longrightarrow} 0$ and

$$\widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H_n^m)}{\Phi^{m\prime}(H_n^m)} + \Phi^{m\prime}(H_n^m)Q \right) \leq \mu$$

for every n. It follows that

$$\Omega^m(Q, H_n^m) \geq \frac{\sum_{j \in f} h_j(r_j(\mu))}{H_n^m} \underset{n \to \infty}{\longrightarrow} \infty.$$

By the same token, if $\lim_{p_j \to \infty} h_j(p_j) > 0$ for some $j \in m$, then $\Omega^m(Q, H^m)$ is bounded below by $\lim_{p_j \to \infty} h_j(p_j)/H^m$, and therefore tends to $+\infty$ as H^m tends to zero.

Finally, assume that $\lim_{p_j \to \infty} h_j(p_j) = 0$ for every $j \in m$, and $\widetilde{m}^f \left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m)Q \right) \underset{H^m \to 0}{\longrightarrow} \bar{\mu}^f$ for every $f \in m$. Note that

$$\begin{aligned}
\Omega^m(Q, H^m) &= \sum_{f \in m} \frac{1}{H^m} \sum_{j \in f} h_j, \\
&= \sum_{f \in m} \frac{1}{H^m} \frac{\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m)Q}{\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m)Q} \sum_{j \in f} h_j, \\
&= \sum_{f \in m} \frac{\widetilde{m}^f - 1}{\widetilde{m}^f} \frac{1}{\sum_{j \in f} \gamma_j} \frac{1}{\vartheta^m(H^m) + H^m \Phi^{m\prime}(H^m)Q} \left(\sum_{j \in f} h_j \right), \\
&= \frac{1}{\vartheta^m(H^m) + H^m \Phi^{m\prime}(H^m)Q} \sum_{f \in m} \frac{\widetilde{m}^f - 1}{\widetilde{m}^f} \frac{\sum_{j \in f} h_j}{\sum_{j \in f} \gamma_j}.
\end{aligned}$$

By Lemma XV, the term $\frac{1}{\vartheta^m(H_n^m) + H_n^m \Phi^{m\prime}(H_n^m)Q}$ tends to $\lim_{H^m \to 0} 1/\vartheta^m(H^m)$, which is strictly greater than 1. Moreover, by Lemma XV, for every firm f, the term $\frac{\widetilde{m}^f - 1}{\widetilde{m}^f} \frac{\sum_{j \in f} h_j}{\sum_{j \in f} \gamma_j}$ tends to a limit which is greater or equal to 1. It follows that $\lim_{H^m \to 0} \Omega^m(Q, H^m) > 1$. Therefore, equation (xxxi) has a solution.

We now prove that the solution is unique. We do so by showing that $\Omega^m(Q, \cdot)$ is strictly decreasing. Let $0 < H^m < H^{m\prime}$. Put

$$X = \frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m)Q \text{ and } X' = \frac{-\Phi^{m\prime\prime}(H^{m\prime})}{\Phi^{m\prime}(H^{m\prime})} + \Phi^{m\prime}(H^{m\prime})Q.$$

Suppose first that $X \leq X'$. Then,

$$\begin{aligned}
\Omega^m(Q, H^m) &= \frac{1}{H^m} \sum_{f \in m} \sum_{j \in f} h_j(r_j(\widetilde{m}^f(X))), \\
&> \frac{1}{H^{m\prime}} \sum_{f \in m} \sum_{j \in f} h_j(r_j(\widetilde{m}^f(X))), \\
&\geq \frac{1}{H^{m\prime}} \sum_{f \in m} \sum_{j \in f} h_j(r_j(\widetilde{m}^f(X'))), \\
&= \Omega^m(Q, H^{m\prime}),
\end{aligned}$$

where the third line follows by Lemma XXII.

Assume instead that $X > X'$. Then,

$$\begin{aligned}
\Omega^m(Q, H^m) &= \frac{1}{\vartheta^m(H^m) + H^m \Phi^{m\prime}(H^m) Q} \sum_{f \in m} \frac{\widetilde{m}^f(X) - 1}{\widetilde{m}^f(X)} \frac{\sum_{j \in f} h_j(r_j(\widetilde{m}^f(X)))}{\sum_{j \in f} \gamma_j(r_j(\widetilde{m}^f(X)))}, \\
&> \frac{1}{\vartheta^m(H^{m\prime}) + H^{m\prime} \Phi^{m\prime}(H^{m\prime}) Q} \sum_{f \in m} \frac{\widetilde{m}^f(X) - 1}{\widetilde{m}^f(X)} \frac{\sum_{j \in f} h_j(r_j(\widetilde{m}^f(X)))}{\sum_{j \in f} \gamma_j(r_j(\widetilde{m}^f(X)))}, \\
&> \frac{1}{\vartheta^m(H^{m\prime}) + H^{m\prime} \Phi^{m\prime}(H^{m\prime}) Q} \sum_{f \in m} \frac{\widetilde{m}^f(X') - 1}{\widetilde{m}^f(X')} \frac{\sum_{j \in f} h_j(r_j(\widetilde{m}^f(X')))}{\sum_{j \in f} \gamma_j(r_j(\widetilde{m}^f(X')))}, \\
&= \Omega^m(Q, H^{m\prime}),
\end{aligned}$$

where the second line follows by Lemma XV and Assumption iii–(h), and the third line follows from Lemmas VII–IX and Assumptions iii–(f) and (g).

Hence, $\Omega^m(Q, \cdot)$ is strictly decreasing, and equation (xxxi) has a unique solution. The continuity of the solution $H^m(Q)$ can then be established by using the same argument as in the proof of Lemma I. Since $m^f(Q) = \widetilde{m}^f\left(\frac{-\Phi^{m\prime\prime}(H^m)}{\Phi^{m\prime}(H^m)} + \Phi^{m\prime}(H^m)Q\right)$ is the composition of two continuous functions, that function is continuous as well.

Finally, we derive the monotonicity properties of $H^m(\cdot)$ and $m^f(\cdot)$. Let $0 < Q < Q'$. Then, by monotonicity of $\widetilde{m}^f$ for every f, we have that $\Omega^m(Q, H^m) > \Omega^m(Q', H^m)$. Since Ω^m is strictly decreasing in H^m, this implies that $H^m(Q) > H^m(Q')$. Assume for a contradiction that

$$X \equiv \frac{-\Phi^{m\prime\prime}(H^m(Q))}{\Phi^{m\prime}(H^m(Q))} + \Phi^{m\prime}(H^m(Q))Q \geq \frac{-\Phi^{m\prime\prime}(H^m(Q'))}{\Phi^{m\prime}(H^m(Q'))} + \Phi^{m\prime}(H^m(Q'))Q' \equiv X'.$$

Then,

$$H^m(Q) = \sum_{f \in m} \sum_{j \in f} h_j\left(r_j(\widetilde{m}^f(X))\right),$$

$$
\begin{aligned}
&\leq \sum_{f\in m}\sum_{j\in f} h_j\left(r_j(\widetilde{m}^f(X'))\right),\\
&= H^m(Q'),
\end{aligned}
$$

which is a contradiction. Hence, $X < X'$, and, for every $f \in m$,

$$
m^f(Q) = \widetilde{m}^f(X) < \widetilde{m}^f(X') = m^f(Q'). \qquad \square
$$

We can finally solve the outer fixed point problem. Define

$$
\Omega(\Phi) = \frac{1}{\Phi}\left(\Phi^0 + \sum_{m\in\mathcal{M}} \Phi^m\left(H^m\left(\frac{-\Psi''(\Phi)}{\Psi'(\Phi)}\right)\right)\right).
$$

Lemma XXIV. *There exists a unique Φ^* such that $\Omega(\Phi^*) = 1$. Moreover, Ω is strictly decreasing.*

Proof. We first show that a solution exists. Ω is continuous. Moreover, $m^f(\cdot)$ is bounded below by 1 for every $m \in \mathcal{M}$ and $f \in m$. Hence,

$$
\Omega(\Phi) \leq \frac{1}{\Phi}\left(\Phi^0 + \sum_{m\in M} \Phi^m\left(\sum_{f\in m}\sum_{j\in f} h_j(r_j(1))\right)\right) \underset{\Phi\to\infty}{\longrightarrow} 0.
$$

If $l^n = \lim_{H^n\to 0}\Phi^n(H^n) > 0$ for some $n \in \mathcal{M}$, or $l_j = \lim_{p_j\to\infty} h_j(p_j) > 0$ for some $j \in m \in \mathcal{M}$, or $\Phi^0 > 0$, then $\Omega(\Phi) \geq l^n/\Phi$, or $\Omega(\Phi) \geq l_j/\Phi$, or $\Omega(\Phi) \geq \Phi^0/\Phi$ for every $\Phi > 0$. Hence, $\lim_{\Phi\to 0}\Omega(\Phi) = \infty$. Assume instead that $\Phi^0 = 0$, $\lim_{H^n\to 0}\Phi^n(H^n) = 0$, and $\lim_{p_j\to\infty} h_j(p_j) = 0$ for every $n \in \mathcal{M}$ and $j \in \mathcal{N}$.

We distinguish two cases. Assume first that $\frac{-\Psi''(\Phi)}{\Psi'(\Phi)}$ does not go to infinity as Φ goes to 0. There exist a sequence $(\tilde{\Phi}^n)_{n\geq 0}$ and an upper bound $M > 0$ such that $\tilde{\Phi}^n \underset{n\to\infty}{\longrightarrow} 0$ and

$$
Q^n = \frac{-\Psi''(\tilde{\Phi}^n)}{\Psi'(\tilde{\Phi}^n)} < M, \quad \forall n \geq 0.
$$

Hence, by monotonicity of $H^m(\cdot)$, $H^m(Q^n) > H^m(M)$, for every $m \in \mathcal{M}$. Therefore,

$$
\Omega(\tilde{\Phi}^n) \geq \frac{1}{\tilde{\Phi}^n}\sum_{m\in\mathcal{M}} H^m(M) \underset{n\to\infty}{\longrightarrow} \infty.
$$

Hence, $\Omega(\Phi) > 1$ for some $\Phi > 0$.

Next, assume instead that $\frac{-\Psi''(\Phi)}{\Psi'(\Phi)}$ does go to infinity as Φ goes to 0. Then, there exists a strictly decreasing sequence $(\tilde{\Phi}^n)_{n\geq 0}$ such that $(Q^n)_{n\geq 0} = \left(\frac{-\Psi''(\tilde{\Phi}^n)}{\Psi'(\tilde{\Phi}^n)}\right)_{n\geq 0}$ is non-decreasing, and $Q^n \underset{n\to\infty}{\longrightarrow} \infty$. The monotonicity properties derived in Lemma XXIII imply that, for every $m \in \mathcal{M}$ and $f \in m$, the sequences $(H^m(Q^n))_{n\geq 0}$ and $\left(m^f(Q^n)\right)_{n\geq 0}$ are respectively non-increasing and non-decreasing. Those sequences therefore have limits. It is then straightforward to use equations (xxviii) and (xxix) to show that $\lim_{n\to\infty} H^m(Q^n) = 0$ and $\lim_{n\to\infty} m^f(Q^n) = \bar{\mu}^f$.

Note that

$$\begin{aligned}
\Omega(\tilde{\Phi}^n) &= \frac{Q^n}{\eta(\tilde{\Phi}^n)} \sum_{m\in\mathcal{M}} \Phi^m(H^m(Q^n)), \\
&= \frac{Q^n}{\eta(\tilde{\Phi}^n)} \sum_{m\in\mathcal{M}} \frac{1}{\varphi^m(H^m(Q^n))} H^m(Q^n)\Phi^{m\prime}(H^m(Q^n)), \\
&= \frac{Q^n}{\eta(\tilde{\Phi}^n)} \sum_{m\in\mathcal{M}} \frac{1}{\varphi^m(H^m(Q^n))} \Phi^{m\prime}(H^m(Q^n)) \sum_{f\in m}\sum_{j\in f} h_j(r_j(m^f(Q^n))), \\
&= \frac{1}{\eta(\tilde{\Phi}^n)} \sum_{m\in\mathcal{M}} \frac{1}{\varphi^m(H^m(Q^n))} \sum_{f\in m} \Phi^{m\prime}(H^m(Q^n))Q^n \sum_{j\in f} h_j(r_j(m^f(Q^n))), \\
&= \frac{1}{\eta(\Phi^n)} \sum_{m\in M} \frac{1}{\varphi^m(H^m(Q^n))} \sum_{f\in m} \left(\frac{m^f(Q^n)-1}{m^f(Q^n)} \frac{\sum_{j\in f} h_j(r_j(m^f(Q^n)))}{\sum_{j\in f} \gamma_j(r_j(m^f(Q^n)))} \right. \\
&\quad \left. - \sum_{j\in f} h_j(r_j(m^f(Q^n))) \frac{-\Phi^{m\prime\prime}(H^m(Q^n))}{\Phi^{m\prime}(H^m(Q^n))} \right), \\
&= \frac{1}{\eta(\Phi^n)} \sum_{m\in M} \frac{1}{\varphi^m(H^m(Q^n))} \left(\left(\sum_{f\in m} \frac{m^f(Q^n)-1}{m^f(Q^n)} \frac{\sum_{j\in f} h_j(r_j(m^f(Q^n)))}{\sum_{j\in f} \gamma_j(r_j(m^f(Q^n)))} \right) - \vartheta^m(H^m(Q^n)) \right), \\
&\geq \sum_{m\in M} \frac{1}{\varphi^m(H^m(Q^n))} \left(\left(\sum_{f\in m} \frac{m^f(Q^n)-1}{m^f(Q^n)} \frac{\sum_{j\in f} h_j(r_j(m^f(Q^n)))}{\sum_{j\in f} \gamma_j(r_j(m^f(Q^n)))} \right) - \vartheta^m(H^m(Q^n)) \right),
\end{aligned}$$

where we have used equation (xxix) to obtain the fifth line, and Lemma XV to obtain the last line. Since $m^f(Q^n) \underset{n\to\infty}{\longrightarrow} \bar{\mu}^f$ and $H^m(Q^n) \underset{n\to\infty}{\longrightarrow} 0$ for every m and f, we can use Lemma XV to conclude that the expression in the last line has a limit as n tends to infinity, and that this limit is bounded below by

$$\sum_{m\in M} \frac{\left(\sum_{f\in m} 1\right) - \lim_{H^m\to 0} \vartheta^m(H^m)}{\lim_{H^m\to 0} \varphi^m(H^m)},$$

which, by Lemma XV–(f), and since there are at least two firms in the industry, is strictly greater than 1. It follows that the equation $\Omega(\Phi) = \Phi$ has a solution.

To prove uniqueness, we show that Ω is strictly decreasing. Let $\Phi' > \Phi > 0$. Put $Q = \frac{-\Psi''(\Phi)}{\Psi'(\Phi)}$ and $Q' = \frac{-\Psi''(\Phi')}{\Psi'(\Phi')}$. If $Q' \geq Q$, then

$$\Omega(\Phi) = \frac{1}{\Phi}\left(\Phi^0 + \sum_{m\in M} \Phi^m(H^m(Q)) \right),$$

$$
\begin{aligned}
&> \frac{1}{\Phi'}\left(\Phi^0 + \sum_{m\in M} \Phi^m(H^m(Q))\right), \\
&\geq \frac{1}{\Phi'}\left(\Phi^0 + \sum_{m\in M} \Phi^m(H^m(Q'))\right), \\
&= \Omega(\Phi').
\end{aligned}
$$

where we have used the monotonicity of $H^m(\cdot)$ to obtain the third line. If instead $Q' < Q$, then, $H^m(Q) < H^m(Q')$, and $m^f(Q) > m^f(Q')$. It follows that

$$
\begin{aligned}
\Omega(\Phi) &= \frac{\Phi^0}{\Phi} + \frac{1}{\eta(\Phi)} \sum_{m\in M} \frac{1}{\varphi^m(H^m(Q))} \left(\left(\sum_{f\in m} \frac{m^f(Q)-1}{m^f(Q)} \frac{\sum_{j\in f} h_j(r_j(m^f(Q)))}{\sum_{j\in f}\gamma_j(r_j(m^f(Q)))}\right) - \vartheta^m(H^m(Q))\right), \\
&> \frac{\Phi^0}{\Phi'} + \frac{1}{\eta(\Phi')} \sum_{m\in M} \frac{1}{\varphi^m(H^m(Q'))} \left(\left(\sum_{f\in m} \frac{m^f(Q')-1}{m^f(Q')} \frac{\sum_{j\in f} h_j(r_j(m^f(Q')))}{\sum_{j\in f}\gamma_j(r_j(m^f(Q')))}\right) - \vartheta^m(H^m(Q'))\right), \\
&= \Omega(\Phi'),
\end{aligned}
$$

where we have used the monotonicity properties of φ^m, ϑ^m, η (Assumptions iii–(b), (h) and (i)), and $\mu^f \mapsto \frac{\mu^f-1}{\mu^f}\frac{\sum_{j\in f} h_j(r_j(\mu^f))}{\sum_{j\in f}\gamma_j(r_j(\mu^f))}$ (Lemmas VII–IX and Assumptions iii–(f) and (g)) to obtain the second line. (Recall that the term in the sum over m is proportional to the contribution of nest m to the industry aggregator, and is therefore strictly positive.) Hence, Ω is strictly decreasing. □

We can conclude:

Theorem III. *Let $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}}, \Phi^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying Assumption iii. The pricing game has a unique equilibrium. The equilibrium aggregator level Φ^* is the unique fixed point of the aggregate fitting-in function. In equilibrium, firm $f \in n$ sets a ι-markup of $\mu^{f*} = m^f\left(\frac{-\Psi''(\Phi^*)}{\Psi'(\Phi^*)}\right)$, and earns a profit of*

$$
(\mu^{f*}-1)\frac{(\Phi^{n\prime}\Psi')^2}{(\Phi^{n\prime})^2(-\Psi'') + (-\Phi^{n\prime\prime})\Psi'},
$$

where the function Ψ and its derivatives are evaluated at

$$
\Phi^0 + \sum_{m\in\mathcal{M}} \Phi^m\left(\sum_{g\in m}\sum_{j\in g} h_j\left(r_j\left(m^g\left(\frac{-\Psi''(\Phi^*)}{\Psi'(\Phi^*)}\right)\right)\right)\right),
$$

and the function Φ^n and its derivatives are evaluated at

$$
\sum_{g\in n}\sum_{j\in g} h_j\left(r_j\left(m^g\left(\frac{-\Psi''(\Phi^*)}{\Psi'(\Phi^*)}\right)\right)\right).
$$

The equilibrium price of product $j \in f$ *is* $r_j\left(m^f\left(\frac{-\Psi''(\Phi^*)}{\Psi'(\Phi^*)}\right)\right)$.

VIII.3 Discussion

Examples. Examples of functional forms satisfying Assumptions iii–(a), (f) and (g) were already given in SectionVI.2. Examples of Φ^m functions satisfying Assumptions iii–(b), (d) and (h) include $\Phi^m(H^m) = \beta(H^0 + H^m)^\alpha$, where $\beta > 0$, $H^0 \geq 0$ and $\alpha \in (0, 1]$ are parameters. Examples of Ψ functions satisfying Assumptions iii–(c), (e), and (i) include $\Psi(\Phi) = \beta \log(\Phi + \Phi^0)$ and $\Psi(\Phi) = \beta(\Phi + \Phi^0)^\alpha$, where $\beta > 0$, $\Phi^0 \geq 0$ and $\alpha \in (0, 1)$ are parameters.

Note that nested CES ($h_i(p_i) = a_i p_i^{1-\sigma}$, $\Phi^m(H^m) = \beta^m(H^m)^\alpha$, $\Psi(\Phi) = \log(\Phi + \Phi^0)$) and MNL ($h_i(p_i) = \exp((a_i - p_i)/\lambda)$, $\Phi^m(H^m) = \beta^m(H^m)^\alpha$, $\Psi(\Phi) = \log(\Phi + \Phi^0)$) demands satisfy Assumption iii. Hence, a pricing game with nested CES or MNL demands has a unique equilibrium, provided that the firm partition is a filtration of the nest partition.

On comparative statics and the monotonicity of fitting-in functions. As in the paper, we can study the impact of entry or a unilateral trade liberalization by performing comparative statics on the parameter Φ^0. Suppose that Φ^0 increases to $\Phi^{0\prime} > \Phi^0$. Then, the aggregate share function $\Omega(\cdot)$, defined in Section VIII.2, shifts upward. Since that function is strictly decreasing, it follows that the equilibrium aggregator level Φ^* increases to $\Phi^{*\prime} > \Phi^*$. Hence, it is still the case that consumers benefit from entry and trade liberalization. (Recall from Section VII that consumer surplus is given by $\Psi(\Phi)$.)

We now use the fitting-in function m^f to study the impact of an increase in Φ^0 on firm f's equilibrium behavior. We have shown in Lemma XXIII that m^f is a strictly increasing function of $Q(\Phi) \equiv -\Psi''(\Phi)/\Psi'(\Phi)$. Hence, firm f reacts to the increase in Φ^0 by lowering its ι-markup, reducing the prices of its products (recall that r_j is increasing in μ^f for every j), and broadening its scope if and only if $Q(\Phi^*) > Q(\Phi^{*\prime})$. If Ψ is the logarithm (as in the paper) or a power function, then the function $Q(\cdot)$ is strictly decreasing on $\mathbb{R}_{++}$, and all the firms therefore respond to entry and trade liberalization by lowering their prices and ι-markups and by introducing new products.

It is however easy to construct a function Ψ that satisfies Assumptions iii–(c), (e), and (i), such that the associated function $Q(\cdot)$ is not globally decreasing. An example of such a function is $\Psi(\Phi) = \text{arsinh}(\Phi)$. Note that $\Phi\Psi'(\Phi) = \Phi/\sqrt{1 + \Phi^2}$ is strictly positive and strictly increasing, and $-\Phi\Psi''(\Phi)/\Psi'(\Phi) = \Phi^2/(1+\Phi^2)$ is non-decreasing, so Assumptions iii–(c), (e), and (i) do hold. However, $-\Psi''(\Phi)/\Psi'(\Phi) = \Phi/(1 + \Phi^2)$ is strictly increasing on $(0, 1)$, and strictly decreasing on $(1, \infty)$. With such a function Ψ, the fitting-in function m^f is therefore hump-shaped in Φ for every firm f. Trade liberalization and entry can therefore have a non-monotonic impact on prices, ι-markups, and the set of active products.

More generally, it is straightforward to show, by integrating a second-order differential equation, that the $\mathcal{C}^2$ function $\Psi : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ satisfies Assumptions iii–(c), (e), and (i) if and only if there exist a continuous and non-decreasing function $\eta : \mathbb{R}_{++} \longrightarrow (0, 1]$ and two constants of integration $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$ such that

$$\Psi(x) = \alpha \int_1^x \exp\left(-\int_1^u \frac{\eta(t)}{t} dt\right) du + \beta.$$

The associated function Q is then given by $Q(x) = \eta(x)/x$. Hence, Q is locally strictly increasing if and only if the elasticity of η (locally) strictly exceeds unity.[13]

Finally, we discuss the impact of an increase in Φ^0 on equilibrium profits. The analysis is more involved than in the paper, because a firm's equilibrium profit is no longer equal to its ι-markup minus 1. Assume that $Q(\Phi^*) > Q(\Phi^{*\prime})$. Let $f \in n \in \mathcal{M}$. Recall from Section VIII.2 that firm f's profit can be written as $G^f((p_j)_{j\in f}, H^0, \Upsilon^0)$, where H^0 denotes the contribution of firm f's rivals within nest n to the nest-level sub-aggregator H^n, and Υ^0 is the contribution of firm f's rivals outside nest n (including the outside option Φ^0) to the industry-level aggregator Φ. Since products are substitutes, G^f is strictly decreasing in H^0 and Υ^0. Moreover, since $Q(\Phi^*) > Q(\Phi^{*\prime})$, all the firms respond to the increase in Φ^0 by lowering their ι-markups. It follows that the equilibrium values of H^0 and Υ^0 go up as the value of the outside option Φ^0 increases to $\Phi^{0\prime}$. A standard revealed profitability argument allows us to conclude that firm f's equilibrium profit decreases.

If instead $Q(\Phi^*) < Q(\Phi^{*\prime})$, then firm f may end up benefiting from the fact that, after Φ^0 increases, its rivals in nest n set higher prices. This countervailing effect may end up offsetting the direct negative effect on firm f's profit of the increase in Φ^0. If $n = \{f\}$, i.e., if firm f is the only firm present in nest n, then this countervailing effect does not exist, and firm f unambiguously suffers from the increase in Φ^0. We provide a formal argument below.

We summarize these insights in a proposition:

Proposition XI. *Let $(\Psi, (\Phi^m)_{m\in\mathcal{M}}, (h_j)_{j\in\mathcal{N}}, \Phi^0, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ be a pricing game satisfying Assumption iii. An increase in Φ^0*

- *raises equilibrium consumer surplus,*
- *induces firms to lower their ι-markups and prices, and expand the set of active products if the equilibrium Q decreases,*
- *induces firms to increase their ι-markups and prices, and prune the set of active products if the equilibrium Q increases,*
- *lowers firm f's equilibrium profit if the equilibrium Q decreases, or if firm f has no rival in its nest.*

Proof. All that is left to do is show that, if firm f has no rival in its nest and $Q(\Phi^*) < Q(\Phi^{*\prime})$,

[13]Note that Q cannot be globally increasing, as this would imply that η would eventually leave the interval $(0, 1]$).

then firm f's equilibrium profit decreases as Φ^0 increases. Let

$$
\begin{aligned}
\Pi^{f,mc}(\mu^f) &= \sum_{k\in f}(r_k(\mu^f) - c_k)(-h_k'(r_k(\mu^f)))\Phi^{n\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f))\right), \\
&= \mu^f \sum_{k\in f} \gamma_k(r_k(\mu^f))\Phi^{n\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f))\right)
\end{aligned}
$$

be firm f's profit (up to a multiplicative constant) under monopolistic competition when it sets the ι-markup μ^f. Let $\mu^f \in [1, \mu^f)$ such that $\bar{\mu}^f \neq \bar{\mu}_j$ for every $j \in f$. Let f' be the set of j's in f such that $\bar{\mu}_j > \mu^f$. Then,

$$
\begin{aligned}
\frac{\partial \log \Pi^{f,mc}}{\partial \mu^f} &= \frac{1}{\mu^f} + \frac{\sum_{j\in f'} r_j'(\mu^f)\gamma_j'(r_j(\mu^f))}{\sum_{j\in f'} \gamma_j(r_j(\mu^f))} + \frac{\Phi^{n\prime\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f))\right)}{\Phi^{n\prime}\left(\sum_{j\in f} h_j(r_j(\mu^f))\right)} \sum_{j\in f'} r_j'(\mu^f)h_j'(r_j(\mu^f)), \\
&= \frac{1}{\mu^f} + \frac{\sum_{j\in f'} r_j'(\mu^f)\gamma_j'(r_j(\mu^f))}{\sum_{j\in f'} \gamma_j(r_j(\mu^f))} - \vartheta^n\left(\sum_{j\in f} h_j(r_j(\mu^f))\right)\frac{\sum_{j\in f'} r_j'(\mu^f)h_j'(r_j(\mu^f))}{\sum_{j\in f} h_j(r_j(\mu^f))}, \\
&= \frac{\sum_{j\in f'} r_j'(\mu^f)h_j'(r_j(\mu^f))}{\sum_{j\in f'} \gamma_j(r_j(\mu^f))}\left(\frac{\mu^f - 1}{\mu^f} - \vartheta^n\left(\sum_{j\in f} h_j(r_j(\mu^f))\right)\frac{\sum_{j\in f'} \gamma_j(r_j(\mu^f))}{\sum_{j\in f} h_j(r_j(\mu^f))}\right), \\
&= \underbrace{\frac{\sum_{j\in f'} r_j'(\mu^f)(-h_j'(r_j(\mu^f)))}{\sum_{j\in f'} \gamma_j(r_j(\mu^f))}}_{>0}\left(\vartheta^n\left(\sum_{j\in f} h_j(r_j(\mu^f))\right)\frac{\sum_{j\in f} \gamma_j(r_j(\mu^f))}{\sum_{j\in f} h_j(r_j(\mu^f))} - \frac{\mu^f - 1}{\mu^f}\right),
\end{aligned}
$$

where the third line follows by Lemma E. If $\vartheta^n\left(\sum_{j\in f} h_j(r_j(1))\right) = 0$, then, by Assumption iii–(i), $\vartheta^n\left(\sum_{j\in f} h_j(r_j(\mu^f))\right) = 0$ for every $\mu^f \geq 1$. Hence, $\frac{\partial \log \Pi^{f,mc}}{\partial \mu^f}(\mu^f) < 0$ for every $\mu^f \in (1, \bar{\mu}^f) \setminus \{\bar{\mu}_i\}_{i\in f}$, and $\Pi^{f,mc}$ is strictly decreasing on $[\mu^{f,mc}, \bar{\mu}^f) \equiv [1, \bar{\mu}^f)$. If instead $\vartheta^n\left(\sum_{j\in f} h_j(r_j(1))\right) > 0$, then, for every $\mu^f > 1$,

$$
\begin{aligned}
\frac{\partial \log \Pi^{f,mc}}{\partial \mu^f} = &\frac{\sum_{j\in f'} r_j'(\mu^f)(-h_j'(r_j(\mu^f)))}{\sum_{j\in f'} \gamma_j(r_j(\mu^f))}\frac{\mu^f - 1}{\mu^f} \\
&\times\left(\frac{\mu^f}{\mu^f - 1}\frac{\sum_{j\in f} \gamma_j(r_j(\mu^f))}{\sum_{j\in f} h_j(r_j(\mu^f))}\vartheta^n\left(\sum_{j\in f} h_j(r_j(\mu^f))\right) - 1\right),
\end{aligned}
$$

Using Assumption iii and the argument in the proof of Lemma XX allows us to conclude that there exists a unique $\mu^f \in (1, \bar{\mu}^f)$ such that

$$
\frac{\mu^f}{\mu^f - 1}\frac{\sum_{j\in f} \gamma_j(r_j(\mu^f))}{\sum_{j\in f} h_j(r_j(\mu^f))}\vartheta^n\left(\sum_{j\in f} h_j(r_j(\mu^f))\right) = 1.
$$

Denote this μ^f by $\mu^{f,mc}$. Then, by monotonicity of $\frac{\mu^f}{\mu^f-1}\frac{\sum_{j\in f}\gamma_j(r_j(\mu^f))}{\sum_{j\in f}h_j(r_j(\mu^f))}\vartheta^n\left(\sum_{j\in f}h_j(r_j(\mu^f))\right)$, $\Pi^{f,mc}$ is strictly increasing on $[1,\mu^{f,mc}]$, and strictly decreasing on $[\mu^{f,mc},\bar{\mu}^f)$.

Next, we argue that $m^f(Q) > \bar{\mu}^{f,mc}$ for every $Q > 0$. To see this, note that $m^f(Q)$ is the unique solution of equation

$$\frac{\mu^f-1}{\mu^f}\frac{\sum_{j\in f}h_j(r_j(\mu^f))}{\sum_{j\in f}\gamma_j(r_j(\mu^f))} = \vartheta^n\left(\sum_{j\in f}h_j(r_j(\mu^f))\right) + \left(\sum_{j\in f}h_j(r_j(\mu^f))\right)\Phi^{n\prime}\left(\sum_{j\in f}h_j(r_j(\mu^f))\right)Q,$$

where we have combined equations (xxviii) and (xxix). Moreover, by Lemma XXIII, m^f is strictly increasing. Hence, $m^f(0)\lim_{Q\to 0} = m^f(Q)$ exists, and $\lim_{Q\to 0} m^f(Q) < m^f(Q)$ for every Q. Moreover, $m^f(0)$ satisfies

$$\frac{\mu^f-1}{\mu^f}\frac{\sum_{j\in f}h_j(r_j(\mu^f))}{\sum_{j\in f}\gamma_j(r_j(\mu^f))} = \vartheta^n\left(\sum_{j\in f}h_j(r_j(\mu^f))\right).$$

Hence, $m^f(0) = \mu^{f,mc}$.

Let π^{f*} and μ^{f^*} (resp. $\pi^{f*\prime}$ and $\mu^{f^*\prime}$) be firm f's equilibrium profit and ι-markup when the value of the outside option is Φ^0 (resp. $\Phi^{0\prime}$). Since $Q(\Phi^*) < Q(\Phi^{*\prime})$, we have that $\mu^{f*\prime} > \mu^{f*} > \mu^{f,mc}$. Therefore,

$$\begin{aligned}\pi^{f*} &= \Pi^{f,mc}(\mu^{f*})\Psi'(\Phi^*),\\ &> \Pi^{f,mc}(\mu^{f*})\Psi'(\Phi^{*\prime}),\\ &> \Pi^{f,mc}(\mu^{f*\prime})\Psi'(\Phi^{*\prime}),\\ &= \pi^{f*\prime},\end{aligned}$$

where the third line follows from the fact that $\Pi^{f,mc}$ is strictly decreasing on $(\mu^{f,mc},\bar{\mu}^f)$. □

IX Additive Aggregation and Demand Systems

IX.1 Characterization Result

We have shown in the paper that the demand system (i) gives rise to aggregative pricing games with additive aggregation. A natural question is whether this property extends to a wider class of demand systems.

For the purpose of this section, it is useful to provide a precise definition of aggregative games and demand systems. We say that the $\mathcal{C}^2$ mapping $D:\mathbb{R}^{\mathcal{N}}_{++}\longrightarrow\mathbb{R}^{\mathcal{N}}$ is a quasi-linear demand system if D satisfies Slutsky symmetry ($\partial D_i/\partial p_j = \partial D_j/\partial p_i$ for every i,j) and $\partial D_i/\partial p_j \neq 0$ for every $i \neq j$.[14] Let $\mathcal{G} = (\mathcal{I},(A_i)_{i\in\mathcal{I}},(\pi_i)_{i\in\mathcal{I}})$ be a normal-form game. Suppose that each action space A_i is a cartesian product of intervals. We say that the game $\mathcal{G}$ is

[14]Recall that Slutsky symmetry is necessary for quasi-linear integrability.

aggregative with additive and smooth aggregation if there exist collections of $\mathcal{C}^2$ functions $(\psi_j)_{j\in\mathcal{I}}$ and $(\phi_j)_{j\in\mathcal{I}}$ such that for every $a=(a_j)_{j\in\mathcal{I}}\in\prod_{j\in\mathcal{I}}A_j$ and $i\in\mathcal{I}$,

$$\pi_i(a)=\phi_i\left(a_i,\sum_{j\in\mathcal{I}}\psi_j(a_j)\right).$$

The following proposition provides a complete characterization of the class of demand systems that give rise to aggregative pricing games:

Proposition XII. *Let D be a quasi-linear demand system. Suppose that the set of products $\mathcal{N}$ contains at least three elements. The following assertions are equivalent:*

(i) Any multiproduct-firm pricing game based on D is aggregative with smooth and additive aggregation.

(ii) There exist $\mathcal{C}^3$ functions Ψ, $(g_i)_{i\in\mathcal{N}}$, and $(h_i)_{i\in\mathcal{N}}$ such that

$$D_i(p)=-g_i'(p_i)-h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}}h_j(p_j)\right),\quad \forall i\in\mathcal{N},\ \forall p>>0. \tag{xxxii}$$

Moreover, consumer surplus is given by:

$$V(p)=\sum_{j\in\mathcal{N}}g_j(p_j)+\Psi\left(\sum_{j\in\mathcal{N}}h_j(p_j)\right).$$

Proof. It is obvious that (ii) implies (i). Assume that (i) holds, and consider the pricing game with firm partition $\{\{i\}\}_{i\in\mathcal{N}}$ and zero marginal cost. Since (i) holds, there exist $\mathcal{C}^2$ functions $\phi_i(p_i,H)$ and $h_i(p_i)$ for every i such that, for every $i\in\mathcal{N}$, the profit of firm $\{i\}$ is given by:

$$\Pi^{\{i\}}(p)=\phi_i\left(p_i,\sum_{j\in\mathcal{N}}h_j(p_j)\right)=p_iD_i(p).$$

It follows that

$$D_i(p)=\frac{1}{p_i}\phi_i\left(p_i,\sum_{j\in\mathcal{N}}h_j(p_j)\right)\equiv f_i\left(p_i,\sum_{j\in\mathcal{N}}h_j(p_j)\right),\quad \forall i.$$

Since $\partial D_i/\partial p_j(p)\neq 0$, it follows that $h_i'(p_i)\neq 0$ for every p_i, and $\partial f_i(p_i,H)/\partial H\neq 0$ for every p_i and H.

By Slutsky symmetry, for every $i\neq j$,

$$h_j'\frac{\partial f_i}{\partial H}(p_i,H)=\frac{\partial D_i}{\partial p_j}=\frac{\partial D_j}{\partial p_i}=h_i'\frac{\partial f_j}{\partial H}(p_j,H). \tag{xxxiii}$$

Next, we differentiate the Slutsky condition with respect to p_k, $k \neq i, j$:

$$h'_j h'_k \frac{\partial^2 f_i}{\partial H^2} = h'_i h'_k \frac{\partial^2 f_j}{\partial H^2}.$$

Since $h'_k \neq 0$, it follows that

$$h'_j \frac{\partial^2 f_i}{\partial H^2} = h'_i \frac{\partial^2 f_j}{\partial H^2}. \tag{xxxiv}$$

Next, differentiate the Slutsky condition with respect to p_i:

$$h'_j \frac{\partial^2 f_i}{\partial p_i \partial H} + h'_j h'_i \frac{\partial^2 f_i}{\partial H^2} = h''_i \frac{\partial f_j}{\partial H} + h'^2_i \frac{\partial^2 f_j}{\partial H^2}.$$

Therefore, using equation (xxxiv),

$$h'_j \frac{\partial^2 f_i}{\partial p_i \partial H} = h''_i \frac{\partial f_j}{\partial H}.$$

Next, we use equation (xxxiii) to eliminate $\partial f_j / \partial H$ and h'_j. This yields:

$$\frac{\frac{\partial^2 f_i}{\partial p_i \partial H}(p_i, H)}{\frac{\partial f_i}{\partial H}(p_i, H)} = \frac{h''_i}{h'_i}.$$

The above condition must hold for every (p_i, H) in the domain of f_i. Note that it depends only on p_i and H (and not on p_j for $j \neq i$). Integrating this partial differential equation, we obtain:

$$\frac{\partial f_i}{\partial H}(p_i, H) = h'_i(p_i)\lambda_i(H),$$

where $\lambda_i(H)$ is a constant of integration. Integrating once more, we obtain:

$$f_i(p_i, H) = h'_i(p_i)\Lambda_i(H) + g'_i(p_i),$$

where Λ_i is an anti-derivative of λ_i, and g'_i is a constant of integration. Therefore,

$$D_i(p) = h'_i(p_i)\Lambda_i\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right) + g'_i(p_i), \quad \forall i.$$

Next, we use Slutsky symmetry one more time:

$$h'_i h'_j \Lambda'_i(H) = h'_i h'_j \Lambda'_j(H).$$

Therefore, Λ_i and Λ_j differ by an additive constant, which we can safely ignore (or, rather, incorporate in the g'_i functions). It follows that (ii) holds. □

Proposition XII generalizes Anderson, Erkal, and Piccinin (2013)'s Propositions 4 and

5. Note that the demand system (xxxii) can be viewed as the sum of a monopoly component ($-g_i'(p_i)$) and an IIA component ($-h_i'(p_i)\Psi'(\sum_j h_j(p_j))$). If the monopoly component is equal to zero for every product, then the demand system boils down to $D_i(p) = -h_i'(p_i)\Psi'(\sum_j h_j(p_j))$, which is a special case (without nests) of the class of demand systems introduced in Section VII and analyzed in Section VIII.[15] A special case where the monopoly component is *not* equal to zero is linear demand (in that case, h_j, g_j' and Ψ' are all affine functions).

In the baseline model studied in the paper, the aggregator $H(p) = \sum_{j\in\mathcal{N}} h_j(p_j)$ is a sufficient statistic for consumer surplus. This property also holds true for the more general demand system (xxxii) if and only if $g_i' = 0$ for every i, i.e., if and only if the demand system has the IIA property. If the demand system does not have the IIA property, then consumer surplus is given by $V(p) = G(p) + \Psi(H(p))$, where $G(p) = \sum_{j\in\mathcal{N}} g_j(p_j)$, i.e., consumer surplus depends on the additively separable aggregators $H(p)$ and $G(p)$.

Whether or not the monopoly component is equal to zero, it is easy to show that any pricing game based on the demand system (xxxii) satisfies a generalized version of the common-ι markup property. We do so in the next subsection.

IX.2 The Generalized Common ι-Markup Property

Fix a pricing game based on the demand system (xxxii). Let $f \in \mathcal{F}$ and $i \in f$. Then,

$$\frac{\partial \Pi^f}{\partial p_i} = -h_i'\Psi' - g_i' - (p_i - c_i)(h_i''\Psi' + g_i'') - \sum_{j\in f}(p_j - c_j)h_j'h_i'\Psi''.$$

Therefore, at any optimum,

$$\frac{p_i - c_i}{p_i}\iota_i(p_i) - \frac{g_i'(p_i) + (p_i - c_i)g_i''(p_i)}{h_i'(p_i)\Psi'(H)} = \underbrace{1 + \frac{\Psi''(H)}{\Psi'(H)}\sum_{j\in f}(p_j - c_j)h_j'(p_i)}_{\equiv \mu^f}.$$

Note that the left-hand side of the above condition only depends on p_i and H, whereas the right-hand side, which we call μ^f, is independent of the identity of product i. Therefore, for a given aggregator level H, firm f's optimal strategy can still be summarized by the uni-dimensional sufficient statistic μ^f. Note that the corresponding pricing function r_i now depends on H and μ^f, as in our analysis of quantity competition in Section XI.

Moreover, r_i is independent of H for every product i if and only if $g_i' = 0$. The following assertions are therefore equivalent:

(i) For every firm partition $\mathcal{F}$, the demand system D gives rise to an aggregative pricing game with additive aggregation. Moreover, for any such pricing game, for every product

[15] Recall that nests are handled in Section VIII are handled by making use of sub-aggregators, i.e., by giving up on fully additive aggregation.

i, the pricing function r_i depends only on μ^f.

(ii) D satisfies the IIA property.

(iii) D can be written as

$$D_i(p) = -h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right).$$

As mentioned above, pricing games based on demand systems that have the IIA property are studied in depth in Section VIII.

X General Equilibrium

In this section, we relax the assumption of quasi-linear preferences, and develop a general equilibrium extension of our framework. As in Neary (2003, 2016)'s treatment of general oligopolistic equilibrium, we study a model with a continuum of sectors and a finite number of firms in each sector. The representative consumer's preferences are represented by an indirect utility function which is additively separable across sectors, as in Bertoletti and Etro (2017). The assumption of indirect additive separability implies that demand in a sector depends on prices in other sectors only through the marginal utility of income, which atomistic firms take as given. This property allows us to use the results derived in Section VIII to characterize the set of equilibria of our general oligopolistic equilibrium model.

X.1 The Demand System

There is a continuum of sectors, indexed by $z \in \mathcal{I}$, where $\mathcal{I} = \bigcup_{k=1}^{K} \mathcal{I}^k$ is a finite and disjoint union of compact intervals. For every $k \in \{1, \ldots, K\}$ and $z \in \mathcal{I}^k$, the set of products in sector z is a finite set $\mathcal{N}^k$ containing at least two elements. For every $1 \leq k \leq K$ and $z \in \mathcal{I}^k$, the price of product $j \in \mathcal{N}^k$ in sector z is denoted by $p_j(z) > 0$. The representative consumer's indirect utility at price profile p and income level $y > 0$ is given by:

$$V(p, y) = \sum_{k=1}^{K} \int_{\mathcal{I}^k} \Psi\left(\sum_{j\in\mathcal{N}^k} h_j\left(\frac{p_j(z)}{y}, z\right), z\right) dz,$$

where:

(a) For every $k \in \{1, \ldots, K\}$ and $j \in \mathcal{N}^k$, h_j is a $\mathcal{C}^3$ function from $\mathbb{R}_{++} \times \mathcal{I}^k$ to $\mathbb{R}_{++}$ such that, for every $z \in \mathcal{I}^k$, $h_j(\cdot, z)$ is strictly decreasing and log-convex.

(b) Ψ is a $\mathcal{C}^3$ function from $\mathbb{R}_{++} \times \mathcal{I}$ to $\mathbb{R}$ such that, for every $z \in \mathcal{I}$, $H \mapsto H\partial_1\Psi(H, z)$ is strictly positive and non-decreasing.[16]

[16] In this section, we denote by $\partial_i f$ the derivative of the function f with respect to the ith argument, and by $\partial^2_{ij} f$ the cross-partial derivative of the function f with respect to the ith and jth arguments.

Assumptions (a) and (b) are the counterparts of conditions (a) and (c) in Proposition IX. Moreover, we restrict attention to price vectors p such that, for every $k \in \{1, \ldots, K\}$ and $j \in \mathcal{N}^k$, $z \in \mathcal{I}^k \mapsto p_j(z)$ is continuous.[17] This restriction, together with the smoothness assumptions imposed above, ensures that all the integrals in this section are well defined.

Properties of V. We now check that V has the properties of an indirect utility function. V is clearly homogeneous of degree 0 in price and income, decreasing in prices and increasing in income. We also need to check that V is quasi-convex in (p, y). We first argue that it is enough to check that $V(p, 1)$ is quasi-convex in p. To see this, suppose that $V(p, 1)$ is indeed quasi-convex, and let (p, y), (p', y'), and $\lambda \in (0, 1)$. Note that

$$\begin{aligned} V\left(\lambda p + (1-\lambda)p', \lambda y + (1-\lambda)y'\right) &= V\left(\frac{\lambda p + (1-\lambda)p'}{\lambda y + (1-\lambda)y'}, 1\right), \\ &= V\left(\frac{\lambda y}{\lambda y + (1-\lambda)y'}\frac{p}{y} + \frac{(1-\lambda)y'}{\lambda y + (1-\lambda)y'}\frac{p'}{y'}, 1\right), \\ &\leq \max\left(V\left(\frac{p}{y}, 1\right), V\left(\frac{p'}{y'}, 1\right)\right), \\ &= \max\left(V\left(p, y\right), V\left(p', y'\right)\right). \end{aligned}$$

Hence, quasi-convexity of $V(\cdot, 1)$ implies quasi-convexity of $V(\cdot, \cdot)$.

By Proposition X, for every $k \in \{1, \ldots, K\}$ and $z \in \mathcal{I}^k$, the function $p \in \mathbb{R}_{++}^{\mathcal{N}^k} \mapsto \Psi\left(\sum_{j \in \mathcal{N}^k} h_j(p_j, z), z\right)$ is convex. It follows that $V(\cdot, 1)$ is convex, hence, quasi-convex.

The demand system. Applying Roy's identity, we find the demand for product $i \in \mathcal{N}^k$ in sector $z \in \mathcal{I}^k$:

$$D_i(p, y) = \frac{-\partial_1 h_i\left(\frac{p_i(z)}{y}, z\right)\partial_1 \Psi\left(\sum_{j \in \mathcal{N}^k} h_j\left(\frac{p_j(z)}{y}, z\right), z\right)}{\sum_{k'=1}^K \int_{z' \in \mathcal{I}_{k'}} \left(\sum_{j \in \mathcal{N}^{k'}} \frac{p_j(z')}{y}\left(-\partial_1 h_j\left(\frac{p_j(z')}{y}, z'\right)\right)\right)\partial_1 \Psi\left(\sum_{j \in \mathcal{N}^{k'}} h_j\left(\frac{p_j(z')}{y}, z'\right), z'\right) dz'}.$$

Thus, demand is equal to the reciprocal of an economy-wide aggregate

$$\sum_{k'=1}^K \int_{z' \in \mathcal{I}_{k'}} \left(\sum_{j \in \mathcal{N}^{k'}} \frac{p_j(z')}{y}\left(-\partial_1 h_j\left(\frac{p_j(z')}{y}, z'\right)\right)\right)\partial_1 \Psi\left(\sum_{j \in \mathcal{N}^{k'}} h_j\left(\frac{p_j(z')}{y}, z'\right), z'\right) dz',$$

which atomistic firms cannot affect, times demand under quasi-linear preferences

$$-\partial_1 h_i\left(\frac{p_i(z)}{y}, z\right)\partial_1 \Psi\left(\sum_{j \in \mathcal{N}^k} h_j\left(\frac{p_j(z)}{y}, z\right), z\right).$$

[17] The equilibrium price profile characterized in Section X.3 satisfies this property.

Special cases. Suppose that, for every H and z, $\Psi(H,z) = \alpha(z)\log H$, where $\alpha(\cdot)$ is a strictly positive and smooth function, and that, for every j and z, $h_j(p_j,z) = a_j(z)p_j^{1-\sigma}$, where $\sigma > 1$, and $a_j(\cdot)$ is a strictly positive and smooth function. Then, the demand system boils down to:

$$D_i(p,y) = \frac{\alpha(z)}{\int_{z'\in\mathcal{I}}\alpha(z')dz'}\frac{a_i(z)p_i(z)^{-\sigma}}{\sum_{j\in\mathcal{N}^k}a_j(z)p_j(z)^{1-\sigma}}y, \quad (k\in\{1,\ldots,K\},\ z\in\mathcal{I}^k,\ i\in\mathcal{N}^k).$$

This demand system, which can be derived from the maximization of a direct utility function with a Cobb-Douglas upper tier and a CES lower tier, is used in Hottman, Redding, and Weinstein (2016).

Another special case arises when, for every H and z, $\Psi(H,z) = \alpha(z)H^\beta$, where $\beta \in (0,1)$, and $\alpha(\cdot)$ is a strictly positive and smooth function, and, for every j and z, $h_j(p_j,z) = a_j(z)p_j^{1-\sigma}$, where $\sigma > 1$, and $a_j(\cdot)$ is a strictly positive and smooth function. In that case, the demand system boils down to:

$$D_i(p,y) = \frac{\alpha(z)a_i(z)p_i(z)^{-\sigma}\left(\sum_{j\in\mathcal{N}^k}a_j(z)p_j(z)^{1-\sigma}\right)^{\beta-1}}{\sum_{k'=1}^K\int_{z'\in\mathcal{I}^{k'}}\alpha(z')\left(\sum_{j\in\mathcal{N}^{k'}}a_j(z')p_j(z')^{1-\sigma}\right)^\beta dz'}y, \quad (1\le k\le K,\ z\in\mathcal{I}^k,\ i\in\mathcal{N}^k).$$

This demand system, which can be derived from the maximization of a direct utility function with CES upper and lower tiers, is used in Atkeson and Burstein (2008) and Edmond, Midrigan, and Xu (2015).

X.2 Multiproduct-Firm Oligopoly Pricing in General Equilibrium

The demand side was already defined in Section X.1. We now describe the supply side, and define the equilibrium concept. For every $k \in \{1,\ldots,K\}$, the set $\mathcal{N}^k$ is partitioned into a set $\mathcal{F}^k$ containing at least two elements. For every $z \in \mathcal{I}^k$, the set of firms present in sector z is indexed by $\mathcal{F}(z) = \mathcal{F}^k$. We assume that each firm is present in only one sector. As in Neary (2003, 2016), this ensures that a firm is big in its own sector (in the sense that it internalizes the impact of its prices on the sector's aggregator), but small in the economy (in the sense that it does not internalize the impact of its prices on the marginal utility of income).

There is a fixed labor supply, $L > 0$. The marginal cost of product $j \in \mathcal{N}^k$ ($k \in \{1,\ldots,K\}$) is $wc_j(z)$, where w is the economy-wide wage rate, and $c_j(z)$ is product j's labor requirement. The representative consumer owns all the firms in the economy. In the following, we normalize total income y to 1.

The profit of firm f operating in sector $z \in \mathcal{I}^k$ is given by:

$$\Pi^f = \frac{1}{\sum_{k'=1}^K\int_{z'\in\mathcal{I}_{k'}}\left(\sum_{j\in\mathcal{N}^{k'}}p_j(z')\left(-\partial_1 h_j\left(p_j(z'),z'\right)\right)\right)\partial_1\Psi\left(\sum_{j\in\mathcal{N}^{k'}}h_j\left(p_j(z'),z'\right),z'\right)dz'}$$

$$\times \sum_{i \in f} (p_i - wc_i(z)) \left(-\partial_1 h_i \left(p_i(z), z\right)\right) \partial_1 \Psi \left(\sum_{j \in \mathcal{N}^k} h_j \left(p_j(z), z\right), z \right).$$

Thus, firm f's profit is equal to the reciprocal of the marginal utility of income, which firm f cannot affect, times firm f's profit in the pricing game with nested demand

$$\Upsilon(z, w) = \left(\Psi(\cdot, z), \Phi^m, (h_j(\cdot, z))_{j \in \mathcal{N}^k}, 0, \mathcal{F}^k, (wc_j(z))_{j \in \mathcal{N}^k}\right),$$

where the nest partition is $\mathcal{M} = \{\mathcal{N}^k\}$, and Φ^m is the identity function. (The notation for pricing games with nested demand was introduced in Section VIII.1.)

An equilibrium is a profile of prices p^* and a wage rate w^* such that: Given the wage rate w^*, for every $k \in \{1, \dots, K\}$ and $z \in \mathcal{I}^k$, $(p_j^*(z))_{j \in \mathcal{N}^k}$ is an equilibrium of the pricing game $\Upsilon(z, w)$; The labor market clears.

We make the following assumptions:

Assumption iv. *(a) For every $k \in \{1, \dots, K\}$ and $j \in \mathcal{N}^k$, h_j is a $\mathcal{C}^3$ function from $\mathbb{R}_{++} \times \mathcal{I}^k$ to $\mathbb{R}_{++}$ such that, for every $z \in \mathcal{I}^k$, $h_j(\cdot, z)$ is strictly decreasing and log-convex.*

(b) Ψ is a $\mathcal{C}^3$ function from $\mathbb{R}_{++} \times \mathcal{I}$ to $\mathbb{R}$ such that, for every $z \in \mathcal{I}$, $H \mapsto H\partial_1\Psi(H, z)$ is strictly positive and non-decreasing.

(c) For every $k \in \{1, \dots, K\}$ and $j \in \mathcal{N}^k$, $c_j(\cdot)$ is continuous.

(d) For every $k \in \{1, \dots, K\}$, $z \in \mathcal{I}^k$, $j \in \mathcal{N}^k$, and $p_j > 0$, $\partial_1 \iota_j(p_j, z) \geq 0$ whenever $\iota_j(p_j, z) > 1$, where $\iota_j(\cdot, z)$ is the absolute value of the elasticity of $-\partial_1 h_j(\cdot, z)$.

(e) For every $k \in \{1, \dots, K\}$, $z \in \mathcal{I}^k$, $f \in \mathcal{F}^k$, and $i, j \in f$, $\bar{\mu}_i(z) = \bar{\mu}_j(z) \equiv \bar{\mu}^f(z)$, where $\bar{\mu}_l(z) \equiv \lim_{p_l \to \infty} \iota_l(p_l, z)$.

(f) For every $k \in \{1, \dots, K\}$, $z \in \mathcal{I}^k$, and $f \in \mathcal{F}^k$, at least one of the following conditions holds:

- $\max_{j \in f} \sup_{p_j > \underline{p}_j(z)} \theta_j(p_j, z) \geq \min_{j \in f} \inf_{p_j > \underline{p}_j(z)} \rho_j(p_j, z)$,
- *$\bar{\mu}^f(z) \leq \mu^*$. Moreover, for every $j \in f$, $\rho_j(\cdot, z)$ is non-decreasing on $(\underline{p}_j(z), \infty)$, and $\lim_{p_j \to \infty} h_j(p_j, z) = 0$,*
- *There exists a function $h^f \in \mathcal{H}^\iota$, a labor requirement level c^f, and a profile of quality shifters $(a_j)_{j \in f} \in \mathbb{R}^f_{++}$, such that $h_j(\cdot, z) = a_j h^f$ and $c_j(z) = c^f$ for every $j \in f$. Moreover, ρ^f is non-decreasing on $(\underline{p}^f, \infty)$,*

where, for every $j \in f$, $\underline{p}_j(z) = \inf\{p_j > 0 : \iota_j(p_j, z) > 1\}$, $\gamma_j(\cdot, z) = (\partial_1 h_j(\cdot, z))^2 / \partial^2_{11} h_j(\cdot, z)$, $\rho_j(\cdot, z) = h_j(\cdot, z)/\gamma_j(\cdot, z)$, $\theta_j(\cdot, z) = \partial_1 h_j(\cdot, z)/\partial_1 \gamma_j(\cdot, z)$, and the threshold μ^* was defined in Section V.2.3.

(g) For every $z \in \mathcal{I}$, $\partial^2_{11}\Psi(\cdot, z) < 0$.

(h) For every $z \in \mathcal{I}$, $H \mapsto H(-\partial^2_{11}\Psi(H, z))/\partial_1\Psi(H, z)$ is non-decreasing.

As shown in the previous section, Assumptions iv–(a) and (b) guarantee that V has the properties of an indirect utility function. Assumptions iv–(d) and (f) are the counterparts of Assumptions iii–(f) and (g). Assumptions iv–(g) and (h) are the counterparts of Assumptions iii–(e) and (i). Assumptions iv–(c) and (e) will allow us to establish the joint continuity of equilibrium prices in the sector index z and the wage rate w.

X.3 Equilibrium analysis

Behavior of equilibrium prices as a function of (z, w). We start by studying equilibrium prices as a function of the sector index $z \in \mathcal{I}^k$ and the wage rate w. Note that, given the wage rate w, the pricing game in sector z satisfies Assumption iii. (The nest partition is simply $\mathcal{M}^k = \{\mathcal{N}^k\}$. The nest function Φ^m is the identity function.) Hence, by Theorem III, there exists a unique equilibrium price vector $(\hat{p}_j(z, w))_{j \in \mathcal{N}^k}$ and a unique equilibrium aggregator level $\hat{H}(z, w)$ in sector z.

We now argue that $(\hat{p}_j(\cdot, \cdot))_{j \in \mathcal{N}^k}$ and $\hat{H}(\cdot, \cdot)$ are both continuous. Let $f \in \mathcal{F}^k$, $j \in f$, and $\mu^f \in (1, \bar{\mu}^f(z))$. Applying the implicit function theorem to the equation

$$\frac{p_j - wc_j(z)}{p_j}\iota_j(p_j, z) = \mu^f,$$

we obtain that the pricing function $r_j(\mu^f, z, w)$ is $\mathcal{C}^1$. Moreover, $\partial_1 r_j > 0$.

The same theorem applied to the equation

$$\frac{\mu^f - 1}{\mu^f} \frac{1}{\sum_{j \in f} \gamma_j(r_j(\mu^f, z, w), z)} = Q$$

implies that the fitting-in function $m^f(Q, z, w)$ is $\mathcal{C}^1$ as well. (See equation (xxx) in Section VIII.2.) Moreover, $\partial_1 m^f > 0$.

Recall that the equilibrium aggregator level $\hat{H}(z, w)$ is pinned down by

$$\Omega(H, z, w) \equiv \frac{1}{H} \sum_{f \in \mathcal{F}^k} \sum_{j \in f} h_j \left(r_j(m^f(Q(H, z), z, w), z, w), z \right) = 1, \tag{xxxv}$$

where $Q(H, z) = -\partial^2_{11}\Psi(H, z)/\partial_1\Psi(H, z)$. In order to apply the implicit function theorem to that equation, we argue that $\partial_1\Omega < 0$. We distinguish two cases. Suppose first that $\partial_1 Q \geq 0$. Then, the derivative of the sum in equation (xxxv) is

$$\sum_{f \in \mathcal{F}^k} \sum_{j \in f} \partial_1 Q \times \partial_1 m^f \times \partial_1 r_j \times \partial_1 h_j \leq 0.$$

Hence, $\partial_1 \Omega < 0$. Suppose instead that $\partial_1 Q < 0$. Note that Ω can be rewritten as

$$
\begin{aligned}
\Omega(H,z,w) &= \frac{1}{HQ(H,z)} \sum_{f \in \mathcal{F}^k} Q(H,z) \sum_{j \in f} h_j \left(r_j(m^f(Q(H,z),z,w),z,w),z \right), \\
&= \frac{1}{\eta(H,z)} \sum_{f \in \mathcal{F}^k} \frac{m^f(Q(H,z),z,w) - 1}{m^f(Q(H,z),z,w)} \frac{\sum_{j \in f} h_j \left(r_j(m^f(Q(H,z),z,w),z,w),z \right)}{\sum_{j \in f} \gamma_j \left(r_j(m^f(Q(H,z),z,w),z,w),z \right)}, \\
&= \frac{1}{\eta(H,z)} \sum_{f \in \mathcal{F}^k} s^f \left(m^f(Q(H,z),z,w),z,w \right),
\end{aligned}
$$

where $\eta(H,z)$ is the absolute value of the elasticity of $\partial_1 \Psi$ with respect to H, and

$$
s^f(\mu^f,z,w) = \frac{\mu^f - 1}{\mu^f} \frac{\sum_{j \in f} h_j \left(r_j(\mu^f,z,w),z \right)}{\sum_{j \in f} \gamma_j \left(r_j(\mu^f,z,w),z \right)}.
$$

By Lemmas VII–IX and Assumption iv–(f), $\partial_1 s^f > 0$. By Assumption iv–(h), $\partial_1 \eta(H,z) \geq 0$. It follows that $\partial_1 \Omega < 0$.

We can therefore apply the implicit function theorem to equation (xxxv) to conclude that $\hat{H}(z,w)$ is $\mathcal{C}^1$. It follows that equilibrium prices

$$
\hat{p}_j(z,w) = r_j \left(m^f \left(Q \left(\hat{H}(z,w),z \right),z,w \right),z,w \right)
$$

are $\mathcal{C}^1$ as well.

Labor demand. The function $\hat{p}_j(\cdot,\cdot)$ and $\hat{H}(\cdot,\cdot)$ allow us to write overall labor demand as as function of the wage rate w:

$$
L^d(w) = \frac{\sum_{k=1}^K \int_{\mathcal{I}^k} \sum_{j \in \mathcal{N}^k} c_j(z) \left(-\partial_1 h_j(\hat{p}_j(z,w),z) \right) \partial_1 \Psi(\hat{H}(z,w),z) dz}{\sum_{k=1}^K \int_{\mathcal{I}^k} \sum_{j \in \mathcal{N}^k} \hat{p}_j(z,w) \left(-\partial_1 h_j(\hat{p}_j(z,w),z) \right) \partial_1 \Psi(\hat{H}(z,w),z) dz}.
$$

Since the integrands are jointly continuous in (z,w) and the domains of integration are compact intervals, $L^d(\cdot)$ is continuous. Moreover, since firms never price below cost, we have that $\hat{p}_j(z,w) \geq w c_j(z)$ for every (z,w). It follows that

$$
\begin{aligned}
L^d(w) &\leq \frac{\sum_{k=1}^K \int_{\mathcal{I}^k} \sum_{j \in \mathcal{N}^k} \frac{1}{w} \hat{p}_j(z,w) \left(-\partial_1 h_j(\hat{p}_j(z,w),z) \right) \partial_1 \Psi(\hat{H}(z,w),z) dz}{\sum_{k=1}^K \int_{\mathcal{I}^k} \sum_{j \in \mathcal{N}^k} \hat{p}_j(z,w) \left(-\partial_1 h_j(\hat{p}_j(z,w),z) \right) \partial_1 \Psi(\hat{H}(z,w),z) dz}, \\
&= \frac{1}{w} \underset{w \to \infty}{\longrightarrow} 0.
\end{aligned}
$$

As a next step, we would like to show that labor demand tends to infinity as the wage rate goes to zero. However, it is easy to show that this is not necessarily the case.[18] To

[18] A similar issue arises in Neary (2016)'s treatment of Cournot oligopoly with a continuum of sectors and

see this, consider the case in which all the sectors are identical, demand is of the MNL type ($h_j(p_j, z) = \mathrm{e}^{\frac{a_j(z)-p_j}{\lambda_j(z)}}$), and there are only two symmetric products with identical marginal costs c in each sector. As w tends to 0, it is easy to show that equilibrium prices converge to those that prevail in a pricing game with MNL demand and 0 marginal cost. Let $\hat{p} > 0$ be that symmetric MNL equilibrium price. Then, as w tends to 0, L^d converges to $c/\hat{p}$, which is finite.

Since L^d does not necessarily tend to infinity as w goes to zero, an equilibrium may fail to exist if L is too high. We now make this statement more precise. Let $\overline{L} = \sup_{w>0} L^d(w) (> 0)$, where $\overline{L}$ may or may not be infinite. The continuity of L^d implies that the range of that function is either $(0, \overline{L})$ or $(0, \overline{L}]$. Hence, an equilibrium exists if $L < \overline{L}$, and does not exist if $L > \overline{L}$.

Equilibrium uniqueness is hard to establish in general, because L^d is not necessarily monotone in w. To see this non-monotonicity, note that the integrand in the denominator in the definition of L^d is equal to industry revenue in a pricing game under quasi-linear preferences. An increase in production costs may or not push the industry closer to industry revenue maximization. Another source of non-monotonicity is that, as we show in Section 3.3 of the paper, $\hat{H}$ does not necessarily decrease when costs increase.

Before turning our attention to special cases, we summarize our results on equilibrium existence in the following proposition:

Proposition XIII. *Fix a model of multiproduct-firm oligopoly pricing in general equilibrium with exogenous labor supply $L > 0$, as defined in Section X.2. Suppose that Assumption iv holds. Then, there exists $\overline{L} \in (0, \infty]$ such that an equilibrium exists if $L < \overline{L}$, and does not exist if $L > \overline{L}$.*

X.4 Special Cases

We now focus on the special case in which $\partial_1 \Psi(H, z) = \alpha(z)/H^{\beta(z)}$, where $\alpha(\cdot)$ and $\beta(\cdot)$ are smooth functions such that $\alpha(z) > 0$ and $0 < \beta(z) \leq 1$ for every z, and $h_j(p_j, z) = a_j(z) p_j^{1-\sigma(z)}$, where $a_j(\cdot)$ and $\sigma(\cdot)$ are smooth functions such that $a_j(z) > 0$ and $\sigma(z) > 1$ for every z. Note that, in the case where β and σ do not vary across sector, the demand system reduces to the one in Hottman, Redding, and Weinstein (2016) (if $\beta = 1$), or in Atkeson and Burstein (2008) and Edmond, Midrigan, and Xu (2015) (if $\beta < 1$). (See the discussion at the end of Section X.1.)

The labor demand function L^d can then be simplified as follows:

$$L^d(w) = \frac{\sum_{k=1}^{K} \int_{\mathcal{I}^k} \alpha(z)(\sigma(z) - 1) \sum_{j \in \mathcal{N}^k} \frac{c_j(z)}{\hat{p}_j(z,w)} \frac{h_j(\hat{p}_j(z,w),z)}{\hat{H}(z,w)^{\beta(z)}} dz}{\sum_{k=1}^{K} \int_{\mathcal{I}^k} \alpha(z)(\sigma(z) - 1) \hat{H}(z,w)^{1-\beta(z)} dz}.$$

linear demand. In his framework, if labor supply is too high, then the market-clearing wage ends up being negative.

We now argue that, for every w, z, and j, $\hat{p}_j(z,w) = w\hat{p}_j(z,1)$. Since $r_j(\mu^f, z, w) = \frac{\sigma(z)}{\sigma(z)-\mu^f} w c_j(z)$ for every j, all we need to do is show that the equilibrium profile of ι-markups in sector z is independent of w. Recall that $(\mu^f)_{f\in\mathcal{F}^k}$ is an equilibrium profile of ι-markups in sector z if and only if, for every firm f,

$$\frac{\mu^f - 1}{\mu^f} = \sum_{j\in f} \gamma_j(r_j(\mu^f, z, w), z) \frac{-\partial^2_{11}\Psi\left(\sum_{g\in\mathcal{F}^k}\sum_{i\in g} h_i(r_i(\mu^g, z, w), z)\right)}{\partial_1\Psi\left(\sum_{g\in\mathcal{F}^k}\sum_{i\in g} h_i(r_i(\mu^g, z, w), z)\right)}.$$

Given the functional form assumptions made above, this is equivalent to

$$\begin{aligned}
\frac{\mu^f - 1}{\mu^f} &= \beta(z)\frac{\sigma(z)-1}{\sigma(z)} \frac{\sum_{j\in f} h_j(r_j(\mu^f, z, w), z)}{\sum_{g\in\mathcal{F}^k}\sum_{i\in g} h_i(r_i(\mu^g, z, w), z)}, \\
&= \beta(z)\frac{\sigma(z)-1}{\sigma(z)} \frac{\sum_{j\in f}\left(\frac{\sigma(z)}{\sigma(z)-\mu^f} w c_j(z)\right)^{1-\sigma(z)}}{\sum_{g\in\mathcal{F}^k}\sum_{i\in g}\left(\frac{\sigma(z)}{\sigma(z)-\mu^g} w c_i(z)\right)^{1-\sigma(z)}}, \\
&= \beta(z)\frac{\sigma(z)-1}{\sigma(z)} \frac{\sum_{j\in f}\left(\frac{\sigma(z)}{\sigma(z)-\mu^f} c_j(z)\right)^{1-\sigma(z)}}{\sum_{g\in\mathcal{F}^k}\sum_{i\in g}\left(\frac{\sigma(z)}{\sigma(z)-\mu^g} c_i(z)\right)^{1-\sigma(z)}}.
\end{aligned}$$

Hence, $(\mu^f)_{f\in\mathcal{F}^k}$ is an equilibrium profile of ι-markups in sector z when the wage rate is w if and only if it is an an equilibrium profile of ι-markups in sector z when the wage rate is 1. This proves our claim that $\hat{p}_j(z,w) = w\hat{p}_j(z,1)$ for every j and z.

L^d therefore simplifies to:

$$L^d(w) = \frac{1}{w}\frac{\sum_{k=1}^K \int_{\mathcal{I}^k} \alpha(z)(\sigma(z)-1)\sum_{j\in\mathcal{N}^k} \frac{c_j(z)}{\hat{p}_j(z,1)} \frac{h_j(\hat{p}_j(z,1),z)}{\hat{H}(z,1)^{\beta(z)}} w^{(1-\sigma(z))(1-\beta(z))} dz}{\sum_{k=1}^K \int_{\mathcal{I}^k} \alpha(z)(\sigma(z)-1)\hat{H}(z,1)^{1-\beta(z)} w^{(1-\sigma(z))(1-\beta(z))} dz}.$$

Define

$$m \equiv \min_{1\le k\le K}\min_{j\in\mathcal{N}^k}\min_{z\in\mathcal{I}^k} \frac{c_j(z)}{\hat{p}_j(z,1)}.$$

By continuity and compactness, the minimum exists, and is strictly positive. Hence,

$$\begin{aligned}
L^d(w) &\geq \frac{1}{w}\frac{\sum_{k=1}^K \int_{\mathcal{I}^k} \alpha(z)(\sigma(z)-1)\sum_{j\in\mathcal{N}^k} m \frac{h_j(\hat{p}_j(z,1),z)}{\hat{H}(z,1)^{\beta(z)}} w^{(1-\sigma(z))(1-\beta(z))} dz}{\sum_{k=1}^K \int_{\mathcal{I}^k} \alpha(z)(\sigma(z)-1)\hat{H}(z,1)^{1-\beta(z)} w^{(1-\sigma(z))(1-\beta(z))} dz}, \\
&= \frac{m}{w} \underset{w\to 0^+}{\longrightarrow} \infty.
\end{aligned}$$

Hence, using the notation introduced in the statement of Proposition XIII, $\overline{L} = \infty$, and an equilibrium always exists.

Equilibrium uniqueness seems harder to establish in general. An immediate observation is that, if $(1-\sigma(z))(1-\beta(z))$ does not vary across sector, as in the demand systems used by Atkeson and Burstein (2008), Edmond, Midrigan, and Xu (2015), and Hottman, Redding, and Weinstein (2016), then, L^d is proportional to $1/w$, and therefore strictly decreasing, and uniqueness follows. More generally, it is easy to show that, if

$$\max_{z\in\mathcal{I}}(1-\sigma(z))(1-\beta(z)) \leq 1+\min_{z\in\mathcal{I}}(1-\sigma(z))(1-\beta(z)),$$

i.e., if the preference parameters σ and β do not vary too much across sector, then L^d is strictly decreasing, and uniqueness follows.

XI Quantity Competition

XI.1 The Demand System

We work with the following family of inverse demand systems:

$$P_i(x) = \frac{h_i'(x_i)}{\sum_{j\in\mathcal{N}} h_j(x_j)},$$

where x_j is the output of good j. We assume that $h_i > 0$ and $h_i' > 0$, i.e., products are substitutes. We also assume that $h_i'' < 0$, which ensures that, under monopolistic competition, the inverse demand for product i is strictly decreasing everywhere. This also implies $\partial P_i/\partial x_i < 0$.

The direct utility function associated with this demand system is $U(x) = \log\sum_{j\in\mathcal{N}} h_j(x_j)$. Since $x \mapsto \sum_{j\in\mathcal{N}} h_j(x_j)$ is strictly concave, and the logarithm is strictly increasing and strictly concave, it follows that U is strictly concave.

XI.2 Assumptions and Technical Preliminaries

We make two assumptions on the limits of h_i'. First, we assume that $\lim_0 h_i' = \infty$. This means that, under monopolistic competition, a firm can always make strictly positive profits by supplying a strictly positive quantity. Second, we assume that $\lim_\infty h_i' = 0$. In other words, the monopolistic competition price of good i goes to 0 as x_i tends to infinity.

Moreover, we assume that monopolistic competition inverse demand functions satisfy Marshall's second law of demand: $|\iota_i|$ is non-decreasing for every i, where $\iota_i(x_i) = x_i\frac{h_i''(x_i)}{h_i'(x_i)}$. Since $h_i' > 0$ and $h_i'' < 0$, this means that $\iota_i' \leq 0$.

Next, we use these assumptions to establish a few basic facts about the functions h_i and ι_i. Note first that $\lim_{x_i\to 0} x_i h_i'(x_i) = 0$. To see this, note that, by the fundamental theorem of calculus,

$$h_i(x_i) - h_i(0) = \int_0^{x_i} h_i'(t)dt \geq x_i h_i'(x_i) \geq 0,$$

where the first inequality follows from the fact that $h_i'' < 0$. By the sandwich theorem, it follows that $\lim_{x_i \to 0} x_i h_i'(x_i) = 0$.

Next, let $\bar{\mu}_i = 1 + \lim_{x_i \to 0} \iota_i(x_i)$. Since $\iota_i \leq 0$ and ι_i is monotone, $\bar{\mu}_i$ exists, and $\bar{\mu}_i \leq 1$. Assume for a contradiction that $\bar{\mu}_i \leq 0$. Then, since ι_i is non-increasing, it follows that $\iota_i(x_i) \leq -1$ for every x_i. Therefore,

$$\frac{d}{dx_i}\left(x_i h_i'(x_i)\right) = x_i h_i''(x_i) + h_i'(x_i) \leq 0.$$

Since $\lim_{x_i \to 0} x_i h_i'(x_i) = 0$, it follows that $x_i h_i'(x_i) \leq 0$ for every x_i. Therefore, $h_i' \leq 0$, a contradiction. We conclude that $\bar{\mu}_i \in (0, 1]$ for every i.

XI.3 The Quantity-Setting Game and the Firm's Profit-Maximization Problem

A quantity-setting game is a triple $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$, where $(h_j)_{j \in \mathcal{N}}$ is an inverse demand system, $\mathcal{F}$ is a partition of the set of products, and $(c_j)_{j \in \mathcal{N}}$ is a vector of marginal costs. The profit of firm $f \in \mathcal{F}$ can be written as follows:

$$\Pi^f(x) = \sum_{\substack{j \in f \\ x_j > 0}} \left(\frac{h_j'(x_j)}{\sum_{i \in \mathcal{N}} h_i(x_i)} - c_j \right) x_j.$$

Fix a firm $f \in \mathcal{F}$, and let $(x_j)_{j \in \mathcal{N} \setminus f}$ such that $\sum_{j \in \mathcal{N} \setminus f} h_j(x_j) > 0$. Then, we claim that the maximization problem

$$\max_{(x_j)_{j \in f} \in [0, \infty)^f} \Pi^f \left((x_j)_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right) \tag{xxxvi}$$

has a solution. To see this, note that the assumptions made and the preliminary results derived in Section XI.2 imply that $\Pi^f(\cdot, (x_j)_{j \in \mathcal{N} \setminus f})$ is continuous on $[0, \infty)^f$. Moreover, since products are substitutes and $\lim_{x_i \to \infty} h_i'(x_i) = 0$ for every i, there exists $M > 0$ such that for every $(x_j)_{j \in f} \in [0, \infty)^f$, there exists $(x_j')_{j \in f} \in [0, M]^f$ such that

$$\Pi^f \left((x_j)_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right) < \Pi^f \left((x_j')_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right).$$

Therefore, the sets of solutions of maximization problems (xxxvi) and

$$\max_{(x_j)_{j \in f} \in [0, M]^f} \Pi^f \left((x_j)_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right) \tag{xxxvii}$$

coincide. Since $\Pi^f(\cdot, (x_j)_{j \in \mathcal{N} \setminus f})$ is continuous and $[0, M]^f$ is compact, maximization problem (xxxvii) has a solution.

XI.4 The Additive Constant ι-Markup Property

We start by deriving first-order conditions under the assumption that all products are active. The derivative of firm f's profit with respect to x_k ($k \in f$) is given by:

$$\frac{\partial \pi^f}{\partial x_k} = \frac{h'_k}{H}\left(-\frac{\sum_{j\in f} x_j h'_j}{H} + x_k \frac{h''_k}{h'_k} + \frac{\frac{h'_k}{H} - c_k}{\frac{h'_k}{H}}\right),$$
$$= \frac{h'_k}{H}\left(-\frac{\sum_{j\in f} x_j h'_j}{H} + \iota_k + \frac{P_k - c_k}{P_k}\right),$$

Therefore, if the first-order conditions hold at output vector $(x_k)_{k\in f}$, then, for every $k \in f$,

$$\frac{P_k - c_k}{P_k} + \iota_k = \frac{\sum_{j\in f} x_j h'_j}{H}.$$

Since the right-hand side of the above condition does not depend on k, it follows that an additive form of the constant ι-markup property holds:

$$\frac{P_k - c_k}{P_k} + \iota_k = \frac{P_l - c_l}{P_l} + \iota_l \equiv \mu^f, \quad \forall k, l \in f.$$

Under monopolistic competition, we would have $\mu^f = \frac{P_k - c_k}{P_k} + \iota_k = 0$. Under oligopoly, the firm internalizes its impact on the aggregator, and sets $\mu^f > 0$.

XI.5 Definition and Properties of Output Functions

Fix $H > 0$, and consider the following function:

$$\nu_k(x_k, H) = 1 - c_k \frac{H}{h'_k(x_k)} + \iota_k(x_k)\left(= \frac{P_k - c_k}{P_k} + \iota_k(x_k)\right).$$

ν_k maps an output level and an aggregator level into a ι-markup. Note that, contrary to the price-competition case, ν_k depends on H.

ν_k is differentiable, $\partial \nu_k / \partial x_k < 0$ (due to $h''_k < 0$ and to Marshall's second law of demand), and $\partial \nu_k / \partial H < 0$. By the inverse function theorem, the inverse function $\chi_k(\mu^f, H)$ is well-defined and differentiable, and satisfies $\partial \chi_k / \partial \mu^f < 0$ and $\partial \chi_k / \partial H < 0$. The output function χ_k maps a ι-markup and an aggregator level into an output level. It plays the same role as the pricing function r_k in the paper.

For every $x_k > 0$,

$$\nu_k(x_k, H) < \sup_{\tilde{x}_k > 0} \nu_k(\tilde{x}_k, H) = \bar{\mu}_k.$$

Therefore, if $\mu^f \geq \bar{\mu}_k$, then the ι-markup μ^f is not consistent with product k being sold. We therefore extend χ_k by continuity: $\chi_k(\mu^f, H) = 0$ whenever $\mu^f \geq \bar{\mu}_k$. Denote $\bar{\mu}^f = \max_{j\in f} \bar{\mu}_j$.

XI.6 Definition and Properties of Markup Fitting-In Functions

Next, we use the output functions defined in the previous subsection to reduce firm f's first-order conditions to a uni-dimensional equation:[19]

$$\mu^f = \frac{1}{H}\sum_{j\in f}\chi_j(\mu^f,H)h'_j\left(\chi_j(\mu^f,H)\right). \tag{xxxviii}$$

Since the right-hand side of condition (xxxviii) is strictly positive, we can safely restrict attention to strictly positive μ^f's. Note that, for every $k\in f$ and $\mu^f\in[0,\bar{\mu}_k)$,

$$\iota_k\left(\chi_k(\mu^f,H)\right) = \mu^f + c_k\frac{H}{h'_k\left(\chi_k(\mu^f,H)\right)} - 1 > -1.$$

Therefore, by definition of ι_k,

$$\chi_k(\mu^f,H)h''_k\left(\chi_k(\mu^f,H)\right) + h'_k\left(\chi_k(\mu^f,H)\right) > 0.$$

Combining the above inequality with the fact that $\partial\chi_k/\partial\mu^f < 0$ for every k such that $\bar{\mu}_k > \mu^f$, it follows that the right-hand side of condition (xxxviii) is strictly decreasing in μ^f on interval $(0,\bar{\mu}^f)$, and identically equal to zero on interval $[\bar{\mu}^f,\infty)$. Since the left-hand side is strictly increasing in μ^f, there exists at most one μ^f such that firm f's simplified optimality condition holds.

If $\mu^f \geq \bar{\mu}^f \equiv \max_{k\in f}\bar{\mu}_k$, then the right-hand side of equation (xxxviii) is equal to zero while the left-hand side is strictly positive. If μ^f is equal to zero, then the right-hand side of equation (xxxviii) is strictly positive, and the left-hand side is equal to zero. Therefore, equation (xxxviii) has a unique solution, which we denote by $m^f(H)$. m^f is firm f's markup fitting-in function.

Totally differentiating equation (xxxviii) yields:[20]

$$d\mu^f = -\frac{dH}{H}\mu^f + \frac{1}{H}\sum_{j\in f}\left(\left.\frac{d(x_jh'_j(x_j))}{dx_j}\right|_{x_j=\chi_j}\left(\frac{\partial\chi_j}{\partial\mu^f}d\mu^f + \frac{\partial\chi_j}{\partial H}dH\right)\right).$$

Therefore,

$$m^{f\prime}(H) = \frac{\frac{-m^f}{H} + \frac{1}{H}\sum_{j\in f}\left(\left.\frac{d(x_jh'_j(x_j))}{dx_j}\right|_{x_j=\chi_j}\frac{\partial\chi_j}{\partial H}\right)}{1-\frac{1}{H}\sum_{j\in f}\left(\left.\frac{d(x_jh'_j(x_j))}{dx_j}\right|_{x_j=\chi_j}\frac{\partial\chi_j}{\partial\mu^f}\right)},$$

which is strictly negative, since $\partial\chi_j/\partial\mu^f < 0$ and $\partial\chi_j/\partial H < 0$ for every j.

By monotonicity, $\lim_{H\to 0} m^f(H)$ and $\lim_{H\to\infty} m^f(H)$ exist. We will compute these limits in the next subsection.

[19] If the j-th term of the sum is such that $\bar{\mu}_j \leq \mu^f$, then $\chi_j(\mu^f,H)h'_j(\chi_j(\mu^f,H)) = \lim_{x_j\to 0} x_jh'_j(x_j) = 0$.

[20] To ease notation, we ignore the fact that the sum should only span those j's that satisfy $\chi_j > 0$.

XI.7 Definition and Properties of Output Fitting-In Functions

For every $k \in f$, let $X_k(H) = \chi_k\left(m^f(H), H\right)$. The function $H \mapsto (X_k(H))_{k\in f}$ is firm f's output fitting-in function.

We first argue that $\lim_{H\to\infty} X_k(H)$ exists and is equal to zero for every k. Assume for a contradiction that this is not the case. There exist $k \in f$, $(H^n)_{n\geq 0}$, and $\varepsilon > 0$ such that $H^n \underset{n\to\infty}{\longrightarrow} \infty$ and $X_k(H^n) > \varepsilon$ for every n. By definition of m^f, we also have that

$$\begin{aligned} m^f(H^n) &= 1 - c_k \frac{H^n}{h_k'(X_k(H^n))} + \iota_k(X_k(H^n)), \\ &< 1 - c_k \frac{H^n}{h_k'(\varepsilon)}, \text{ since } X_k(H^n) > \varepsilon,\ h_k'' < 0, \text{ and } \iota_k \leq 0, \\ &\underset{n\to\infty}{\longrightarrow} -\infty. \end{aligned}$$

Therefore, $m^f(H^n)$ is strictly negative for n high enough, a contradiction. Therefore, $\lim_{H\to\infty} X_k(H) = 0$.

Next, we argue that $\lim_{H\to\infty} m^f(H) = 0$. Condition (xxxviii) can be rewritten as follows:

$$m^f(H) = \frac{1}{H} \sum_{j\in f} X_j(H) h_j'\left(X_j(H)\right).$$

Since, for every f, $\lim_{H\to\infty} X_j(H) = 0$ and $\lim_{x_j\to 0} x_j h_j'(x_j) = 0$, it follows that $\lim_{H\to\infty} m^f(H) = 0$.

Next, assume for a contradiction that X_k does not go to zero as H goes to 0 for some k in f. There exist $\varepsilon > 0$ and $(H^n)_{n\geq 0}$ such that $H^n \underset{n\to\infty}{\longrightarrow} 0$ and $X_k(H^n) > \varepsilon$ for every n. Recall that the function $x_k \mapsto x_k h_k'(x_k)$ is strictly increasing on the relevant domain (see Section XI.6). It follows that, for every n,

$$\begin{aligned} m^f(H^n) &= \frac{1}{H^n} \sum_{j\in f} X_j(H^n) h_j'\left(X_j(H^n)\right), \\ &\geq \frac{1}{H^n} X_k(H^n) h_k'\left(X_k(H^n)\right), \\ &\geq \frac{1}{H^n} \varepsilon h_k'(\varepsilon), \\ &\underset{n\to\infty}{\longrightarrow} \infty. \end{aligned}$$

Since m^f is always below unity, this is a contradiction. Therefore, $\lim_{H\to 0} X_k(H) = 0$.

It follows immediately that $\lim_{H\to 0} m^f(H) = \bar{\mu}^f$. As competition intensifies (H goes up), firm f decreases its ι-markup from $\bar{\mu}^f$ (the monopoly case) to 0 (the monopolistic competition limit), and the set of products offered by firm f expands.

By contrast, the output fitting-in function X_k is non-monotonic in H: $X_k(0) = X_k(\infty) = 0$, and $X_k(H) > 0$ for H high enough (if $\bar{\mu}_k < \bar{\mu}^f$, then $X_k = 0$ for H sufficiently low).

XI.8 Definition and Properties of the Aggregate Fitting-In Function

The aggregate fitting-in function is defined as follows:

$$\Gamma(H) = \sum_{f \in \mathcal{F}} \sum_{j \in f} h_j \left(X_j(H) \right).$$

Since $\Gamma(0) = \Gamma(\infty) = \sum_{j \in \mathcal{N}} h_j(0)$ and $\Gamma(H) > \sum_{j \in \mathcal{N}} h_j(0)$ for every $H > 0$, Γ is non-monotonic.

In the following, we first establish the existence of an $H^* > 0$ such that $\Gamma(H^*) = H^*$. If $\lim_{x_k \to 0} h_k(x_k) > 0$ for some $k \in \mathcal{N}$, then this is trivial: Since Γ is continuous, $\Gamma(0) > 0$, and $\Gamma(\infty) < \infty$, existence of a fixed point follows from the intermediate value theorem.

Next, assume that $h_j(0) = 0$ for every j. Note first that, by L'Hospital's rule, for every j,

$$\lim_{x \to 0} \frac{h_j(x)}{x h_j'(x)} = \lim_{x \to 0} \frac{h_j'(x)}{h_j'(x) + x h_j''(x)} = \lim_{x \to 0} \frac{1}{1 + \iota_j(x)} = \frac{1}{\bar{\mu}_k}.$$

To simplify the exposition, assume that $\bar{\mu}^f = \bar{\mu}_k$ for every f and $k \in f$. The case where this assumption is violated can be dealt with as we do in the proof of Lemma J (by taking an H small enough such that all the firms are only supplying their high $\bar{\mu}_k$ products). Take some $\varepsilon > 0$ such that $|\mathcal{F}|(1 - \varepsilon) > 1$. There exists $\hat{H} > 0$ such that $\frac{h_j(X_j(H))}{X_j(H) h_j'(X_j(H))} \geq (1 - \varepsilon) \frac{1}{\bar{\mu}^f}$ for every $H < \hat{H}$, $f \in \mathcal{F}$ and $j \in f$. Moreover, for every $H < \hat{H}$,

$$\begin{aligned}
\frac{\Gamma(H)}{H} &= \sum_{f \in \mathcal{F}} \sum_{j \in f} \frac{h_j(X_j(H))}{H}, \\
&= \sum_{f \in \mathcal{F}} \sum_{j \in f} \frac{X_j(H) h_j'(X_j(H))}{H} \frac{h_j(X_j(H))}{X_j(H) h_j'(X_j(H))}, \\
&\geq (1 - \varepsilon) \sum_{f \in \mathcal{F}} \frac{1}{\bar{\mu}^f} \frac{1}{H} \sum_{j \in f} X_j(H) h_j'(X_j(H)) \\
&= (1 - \varepsilon) \sum_{f \in \mathcal{F}} \frac{1}{\bar{\mu}^f} m^f(H), \text{ by condition (xxxviii)}, \\
&\underset{H \to 0}{\longrightarrow} (1 - \varepsilon) \sum_{f \in \mathcal{F}} 1, \text{ since } \lim_{H \to 0} m^f(H) = \bar{\mu}^f, \\
&= |\mathcal{F}|(1 - \varepsilon), \\
&> 1.
\end{aligned}$$

It follows that $\Gamma(H) > H$ in the neighborhood of zero. The fact that $\lim_{H \to \infty} \Gamma(H) = 0$ and the continuity of Γ give us the existence of a fixed point.

XI.9 Equilibrium Uniqueness and Sufficiency of First-Order Conditions

In the previous subsection, we established the existence of an aggregator level H^* such that $\Gamma(H^*) = H^*$. Since we have not shown that first-order conditions are sufficient for global optimality, we cannot conclude that H^* is an equilibrium aggregator level.

Suppose that the following condition holds:

$$\sum_{j \in f} \left(H X_j'(H) h_j'\left(X_j(H)\right) - h_j\left(X_j(H)\right) \right) < 0, \quad \forall f \in \mathcal{F},\ \forall H > 0. \tag{xxxix}$$

Fix a firm $f \in \mathcal{F}$ and a profile of outputs for firm f's rivals $(x_j)_{j \in \mathcal{N} \setminus f}$ such that $H^0 = \sum_{j \in \mathcal{N} \setminus f} h_j(x_j) > 0$. Define

$$\Omega^f(H, H^0) = \frac{1}{H} \left(H^0 + \sum_{j \in f} h_j(X_j(H)) \right).$$

The first-order conditions associated with firm f's profit-maximization problem hold at output vector $(x_j)_{j \in f}$ if and only if there exists $H > 0$ such that $x_j = X_j(H)$ for every $j \in f$, and $\Omega^f(H, H^0) = 1$. Since $\Omega^f(0, H^0) = \infty$, $\Omega^f(\infty, H^0) = 0$, and $\Omega^f(\cdot, H^0)$ is continuous, there exists $H > 0$ such that $\Omega^f(H, H^0) = 1$. Moreover, for every $H > 0$,

$$\begin{aligned}
\frac{\partial \Omega^f}{\partial H} &= \frac{1}{H^2} \left(\sum_{j \in f} X_j'(H) h_j'(X_j(H)) H - (H^0 + \sum_{j \in f} h_j(X_j(H))) \right), \\
&< \frac{1}{H^2} \sum_{j \in f} \left(H X_j'(H) h_j'(X_j(H)) - h_j(X_j(H)) \right), \\
&< 0, \text{ by condition (xxxix).}
\end{aligned}$$

Therefore, $\Omega^f(\cdot, H^0)$ is strictly decreasing, and there exists a unique $H > 0$ such that $\Omega^f(H, H^0) = 1$. This means that there exists a unique output profile $(\tilde{x}_j)_{j \in f}$ for firm f such that firm f's first-order conditions hold. In Section XI.3, we have shown that firm f's profit maximization problem has a solution $(\hat{x}_j)_{j \in f}$. By necessity, first-order conditions must hold at output profile $(\hat{x}_j)_{j \in f}$. By uniqueness, $(\tilde{x}_j)_{j \in f} = (\hat{x}_j)_{j \in f}$. Therefore, first-order conditions are necessary and sufficient for optimality.

This implies that H is an equilibrium aggregator level if and only if H is a fixed point of the aggregate fitting-in function. Since we have established existence of such a fixed point, it follows that the quantity-setting game has a Nash equilibrium.

In fact, under condition (xxxix), we can even prove that the quantity-setting game has a unique equilibrium. To see this, define $\Omega(H) = \Gamma(H)/H$. Then,

$$\Omega'(H) = \frac{1}{H^2}\left(\sum_{f\in\mathcal{F}}\sum_{j\in f} HX_j'(H)h_j'(X_j(H)) - \sum_{f\in\mathcal{F}}\sum_{j\in f} h_j(X_j(H))\right),$$

which is strictly negative by condition (xxxix). Therefore, the aggregate fitting-in function has a unique fixed point, and the quantity-setting game has a unique equilibrium.

XI.10 The CES Case

In the following, we show that condition (xxxix) holds in the CES case. For every $j \in \mathcal{N}$, let $h_j(x_j) = a_j x_j^{\alpha}$, where $a_j > 0$ is a quality parameter, and $\alpha \in (0,1)$. Clearly, h_j is strictly increasing and strictly concave, $\lim_{x_j\to 0} h_j'(x_j) = \infty$, and $\lim_{x_j\to\infty} h_j'(x_j) = 0$. Moreover, $\iota_j = \alpha - 1$.

Note that, for every firm f,

$$m^f(H) = \frac{1}{H}\sum_{j\in f} X_j(H)h_j'(X_j(H)) = \frac{\alpha}{H}\sum_{j\in f} h_j(X_j(H)).$$

Therefore,

$$m^{f\prime}(H) = \frac{\alpha}{H^2}\sum_{j\in f}\left(HX_j'(H)h_j'(X_j(H)) - h_j(X_j(H))\right).$$

Since $m^{f\prime} < 0$, it follows that $\sum_{j\in f}\left(HX_j'(H)h_j'(X_j(H)) - h_j(X_j(H))\right) < 0$, i.e., condition (xxxix) holds. Therefore, multiproduct-firm quantity-setting games with CES demands have a unique equilibrium.

XI.11 Firm Scope and Industry Competitiveness under Quantity Competition

As already mentioned in Section XI.7, as competition intensifies (H increases), firm f reacts by lowering its ι-markup, and the set of products offered by firm f expands. Hence, under quantity competition, it is still the case that firms respond to an increase in the intensity of competition by adding products.

To illustrate this phenomenon, we consider a simple numerical example, in which a monopolist owning two products, 1 and 2, competes against an outside option $H^0 > 0$. The inverse demand function for product i is given by

$$P_i = \frac{h_i'(x_i)}{h_1(x_1) + h_2(x_2) + H^0},$$

where $h_j(x_j) = x_j^{\alpha_j}$ ($j \in \{1,2\}$), $\alpha_1 = 0.5$, and $\alpha_2 = 0.8$. We set both products' marginal costs equal to 1. We check numerically that the profit maximization problem has a unique

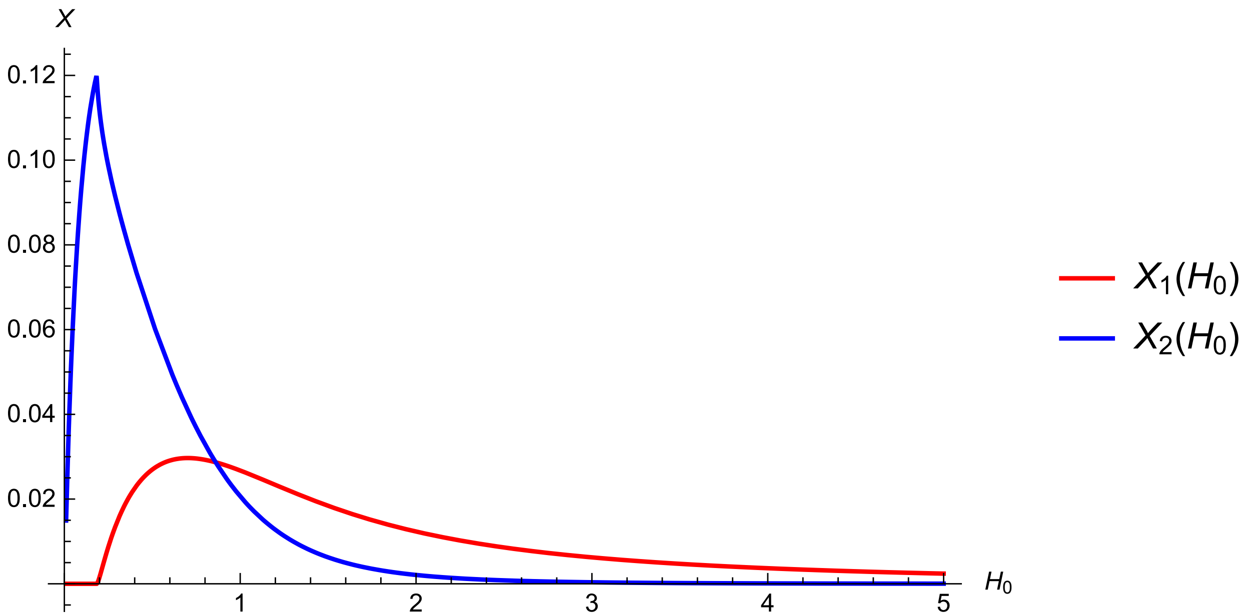

Figure 1: Monopolist's optimal output for products 1 and 2 as a function of H^0

solution, and first-order conditions are sufficient for optimality for every $H^0 > 0$. Figure 1 plots the monopolist's optimal output for products 1 and 2 as a function of H^0. Since $\bar{\mu}_2 > \bar{\mu}_1$, product 2 is always active, whereas product 1 is inactive when H^0 is sufficiently small.

XII Firm Scope and the Intensity of Competition

Our model predicts that multiproduct firms respond to an increase in the intensity of competition by broadening their scope. As shown in Section 3.2, the fitting-in function $m^f(H)$ is strictly decreasing in H, implying that the set of products j in f such that $\bar{\mu}_j > m^f(H)$ expands as H increases. This implies that a shock that shifts the aggregate fitting-in function upward, such as a unilateral trade liberalization or the entry of a new competitor, induces firms to supply more products (Proposition 4).

The intuition is rooted in the IIA property, which implies that, when a firm that has a low market share introduces a new product, that new product mostly cannibalizes sales from the firm's rivals rather than from the firm's other products. Hence, a firm that operates in a highly competitive environment worries little about self-cannibalization effects, and instead has an incentive to flood the market with its products, in order to increase the probability that one of its products is chosen by any given consumer. By contrast, a firm that operates in an environment with little competition has an incentive to withdraw its weaker products (i.e., those products on which the firm earns a low profit conditional on the product being chosen) in order to channel consumers towards its stronger products.

As shown in Section XI, these results extend readily to the case of quantity competition, at least within the class of demand systems we consider. Since the fitting-in function m^f continues to be decreasing in the aggregator level H, the set of active products continues to expand as competition intensifies. Section XI.11 provides a numerical example. The prediction is more nuanced in Section VIII, where we consider richer substitution patterns between the firms' products and the outside option, as captured by the function $\Psi(\cdot)$. As discussed in Section VIII.3, the local monotonicity properties of the fitting-in function $\tilde{m}^f(\cdot) \equiv m^f(Q(\cdot))$ depend on the local behavior of the curvature of $\Psi(\cdot)$, as measured by $Q(\cdot) = -\Psi''(\cdot)/\Psi'(\cdot)$. However, it is easy to show that, since the curvature function Q tends to 0 as the aggregator tends to infinity, firm f's ι-markup tends to 1 as the aggregator tends to infinity, implying that, as we approach the monopolistic competition limit, firm f starts supplying all of its products.

The relationship between firm scope and the intensity of competition has received much attention in the recent international trade literature studying multiproduct firms. Much of that literature has focused on models of monopolistic competition, thereby assuming away the self-cannibalization effects we emphasize. A common finding in those papers is that firms tend to respond to trade liberalization by focusing on their core products, i.e., by supplying fewer products.[21] In models with CES demand and product-level fixed costs (Bernard, Redding, and Schott, 2010, 2011), this is due to the fact that more intense competition reduces variable profits on all products, and therefore makes it harder to cover fixed costs. In models with linear demand, more intense competition chokes out the demand for products sold at a high price (Dhingra, 2013; Mayer, Melitz, and Ottaviano, 2014).

Eckel and Neary (2010) and Eckel, Iacovone, Javorcik, and Neary (2015) develop oligopoly models with multiproduct firms, linear demand, and quantity competition. Despite the presence of the self-cannibalization effect which, as mentioned above, is the key driving force behind our results, they find that firms shed products as competition intensifies.

To understand why their predictions differ from ours, consider the following thought experiment. Suppose that firm f owns two products, i and j, and contemplates whether to supply product i in addition to product j. The demand for product $k \in \{i, j\}$ is given by $D_k(p_i, p_j, H^0)$, where H^0 is a proxy for the intensity of competition. If firm f only sells good j, then it prices that product at $p_j^*(H^0)$, the stand-alone best-response price for that product. That price is pinned down by the first-order condition

$$(p_j - c_j)\frac{\partial D_j}{\partial p_j}(\infty, p_j, H^0) + D_j(\infty, p_j, H^0) = 0.$$

Let $\pi_j^*(H^0)$ be the profit of firm f at that price.

Let $\overline{p}_i(H^0)$ be the lowest price p_i for good i such that, if product j is priced at $p_j^*(H^0)$ and industry competitiveness is H^0, then good i receives no demand. Note that $\overline{p}_i(H^0)$ is infinite in our framework. (In the extension developed in Section IV, $\overline{p}_i(H^0)$ is a strictly positive constant.) One way of finding out whether firm f would find it profitable to supply good i in addition to good j is to ask whether that firm would benefit from setting p_i just below

[21] Qiu and Zhou (2013) and Nocke and Yeaple (2014) derive more nuanced predictions.

$\overline{p}_i(H^0)$, while continuing to price good j at $p_j^*(H^0)$. The marginal profit on good i is given by

$$
\begin{aligned}
\frac{\partial \Pi^f}{\partial p_i}(p_i, p_j^*(H^0)) &= D_i(p_i, p_j^*(H^0)) + (p_i - c_i)\frac{\partial D_i}{\partial p_i}(p_i, p_j^*(H^0)) + (p_j^*(H^0) - c_j)\frac{\partial D_j}{\partial p_i}(p_i, p_j^*(H^0)), \\
&= (p_j^*(H^0) - c_j)\frac{\partial D_j}{\partial p_i}(p_i, p_j^*(H^0)) \left(1 + \frac{D_i(p_i, p_j^*(H^0))}{(p_j^*(H^0) - c_j)\frac{\partial D_j}{\partial p_i}(p_i, p_j^*(H^0))}\right. \\
&\quad \left. + \frac{(p_i - c_i)}{(p_j^*(H^0) - c_j)} \frac{\frac{\partial D_i}{\partial p_i}(p_i, p_j^*(H^0))}{\frac{\partial D_j}{\partial p_i}(p_i, p_j^*(H^0))}\right)
\end{aligned}
$$

Define

$$
\delta(H^0) = \lim_{p_i \to \overline{p}_i(H^0)^-} \left(\frac{(p_i - c_i)}{(p_j^*(H^0) - c_j)} \left| \frac{\frac{\partial D_i}{\partial p_i}(p_i, p_j^*(H^0))}{\frac{\partial D_j}{\partial p_i}(p_i, p_j^*(H^0))} \right| - \frac{D_i(p_i, p_j^*(H^0))}{(p_j^*(H^0) - c_j)\frac{\partial D_j}{\partial p_i}(p_i, p_j^*(H^0))} \right).
$$

(The limit exists in the examples considered below.)

The marginal profit on good i for p_i close enough to $\overline{p}_i(H^0)$ is positive if $\delta(H^0) < 1$, and negative if $\delta(H^0) > 1$. This means that firm f finds it profitable (resp., unprofitable) to supply good i if $\delta(H^0) > 1$ (resp., $\delta(H^0) < 1$). (If $\delta(H^0) = 1$, then the test is inconclusive.) Whether $\delta(H^0)$ is greater or lower than unity depends on the ratio of margins of the two goods (selling good i appears more profitable if a high margin can be set on that good) and on the diversion ratio from i to j (selling good i appears more profitable if that good does not cannibalize good j too much).

We now use the sufficient statistic $\delta(H^0)$ to investigate whether an increase in the intensity of competition raises or lowers the incentives to supply good i. Formally, we ask whether $\delta'(H^0)$ is positive or negative when $\delta(H^0) = 1$.

In our model, we have that

$$
\begin{aligned}
-\frac{p_i - c_i}{p_j - c_j}\frac{\frac{\partial D_i}{\partial p_i}}{\frac{\partial D_j}{\partial p_i}} - \frac{D_i}{(p_j - c_j)\frac{\partial D_j}{\partial p_i}} &= -\frac{p_i - c_i}{p_j - c_j}\frac{h_i'' H - (h_i')^2}{-h_j' h_i'} - \frac{H}{(p_j - c_j)(-h_j')}, \\
&= \frac{1}{(p_j - c_j)(-h_j')}\frac{p_i - c_i}{p_i}\left(\iota_i H + p_i h_i'\right) - \frac{H}{(p_j - c_j)(-h_j')}.
\end{aligned}
$$

Taking the limit as p_i tends to infinity (which is the choke price in our framework), we obtain that

$$
\delta(H^0) = \frac{1}{\pi_j^*(H^0)}\left(\bar{\mu}_i - 1\right).
$$

Since $\pi_j^*(H^0)$ is strictly decreasing in H^0, it follows that $\delta' > 0$. Hence, more intense competition makes it more likely that product i is supplied.

We now turn our attention to the case where demand is linear. The demand for product

$k \in \{1, 2\}$ (when both products are active) is given by

$$D_k = 1 - H^0 - p_k + \gamma p_l, \quad (l \neq k),$$

where $\gamma \in (0, 1)$ is a substitutability parameter. $H^0 > 0$ is a proxy that captures how low rivals' prices are.

Setting D_i equal to zero, we obtain the choke price for product i as a function of p_j and H^0:

$$\overline{p}_i(p_j, H^0) = 1 - H^0 + \gamma p_j.$$

Plugging this choke price into D_j gives us the demand for product j when product i is inactive:

$$\hat{D}_j = (1 + \gamma)\left(1 - H^0 - p_j(1 - \gamma)\right).$$

Solving the profit maximization problem for good j, we obtain the stand-alone best-response price $p_j^*(H^0)$:

$$p_j^*(H^0) = \frac{1}{2}\left(c_j + \frac{1 - H^0}{1 - \gamma}\right).$$

The choke price of good i is therefore given by:

$$\overline{p}_i(H^0) = \overline{p}_i(p_j^*(H^0), H^0) = 1 - H^0 + \frac{1}{2}\gamma\left(c_j + \frac{1 - H^0}{1 - \gamma}\right).$$

We can now compute the sufficient statistic $\delta(H^0)$:

$$\begin{aligned}\delta(H^0) &= \frac{1}{\gamma}\frac{\overline{p}_i(H^0) - c_i}{p_j^*(H^0) - c_j} - 0, \\ &= \frac{1}{\gamma}\frac{1 - H^0 + \frac{1}{2}\gamma\left(c_j + \frac{1-H^0}{1-\gamma}\right) - c_i}{\frac{1}{2}\left(\frac{1-H^0}{1-\gamma} - c_j\right)}.\end{aligned}$$

Thus, whether δ is increasing or decreasing in the neighborhood of $\delta(H^0) = 1$ depends on whether the choke price $\overline{p}_i(H^0)$ decreases faster than the stand-alone best-response price $p_i^*(H^0)$.[22] (The diversion ratio remains constant and equal to γ.) We now compute the corresponding derivative:

$$\begin{aligned}\left.\frac{d\delta}{dH^0}\right|_{\delta(H^0)=1} &= \frac{1}{\gamma}\frac{1}{(p_j^*(H^0) - c_j)^2}\left(\overline{p}_i'(H^0)(p_j^*(H^0) - c_j) - (\overline{p}_i(H^0) - c_i)p_j^{*\prime}(H^0)\right), \\ &= \frac{1}{\gamma}\frac{1}{p_j^*(H^0) - c_j}\left(\overline{p}_i'(H^0) - \gamma p_j^{*\prime}(H^0)\right), \text{ since } \delta(H^0) = 1,\end{aligned}$$

[22] If there is no H^0 such that $\delta(H^0) = 1$, then, regardless of H^0, either the firm always wants to add product i, or it never wants to do so.

$$= \frac{1}{\gamma} \frac{1}{p_j^*(H^0) - c_j} \left(\left(-1 - \frac{1}{2} \frac{\gamma}{1-\gamma} \right) + \frac{1}{2} \frac{\gamma}{1-\gamma} \right),$$
$$= \frac{-1}{\gamma} \frac{1}{p_j^*(H^0) - c_j} < 0.$$

Hence, as H^0 increases, the choke price on good i decreases faster than the stand-alone best-response price on good j, and the firm wants to drop product i.

We can now see why our predictions differ from those in Eckel and Neary (2010) and Eckel, Iacovone, Javorcik, and Neary (2015). In our framework, there is no horse race between the choke price and the stand-alone best-response price, because our choke price remains fixed at $\overline{p}_i = \infty$ (or $\overline{p}_i < \infty$ in the extension studied in Section IV). Instead, what drives our comparative statics is the behavior of the diversion ratio, which is governed by the IIA property. This diversion ratio is constant under linear demand.

XIII Nested CES and MNL Demands: Type Aggregation and Algorithm

In this section, we study a multiproduct-firm pricing game with nested CES or MNL demands, under the assumption that the firm partition $\mathcal{F}$ and the nest partition $\mathcal{M}$ coincide. Under nested CES demand, $\Psi = \log$, $\Phi^f(H^f) = (H^f)^\beta$, and $h_j(p_j) = a_j p_j^{1-\sigma}$, where $\beta \in (0,1]$, $a_j > 0$, and $\sigma > 1$ are parameters. Under nested MNL demand, $\Psi = \log$, $\Phi^f(H^f) = (H^f)^\beta$, and $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda}}$, where $\beta \in (0,1]$, $a_j \in \mathbb{R}$, and $\lambda > 0$ are parameters.[23] Recall that any such pricing game has a unique equilibrium (Theorem III).

This section is organized as follows. In Section XIII.1, we prove the formal equivalence between pricing games with nested CES (resp. MNL) demand and pricing games with CES (resp. MNL) demand. We provide an algorithm for computing equilibrium in Section XIII.2. The proofs are contained in Sections XIII.3 and XIII.4.

XIII.1 Formal Equivalence between Pricing Games with and without Nests

(Nested) CES demand. We first argue that a pricing game with nested CES demand is formally equivalent to a pricing game with CES demand (i.e., where $\beta = 1$). Under nested CES demand, $\iota_j = \sigma$ for every j. Hence, $r_j(\mu^f) = \frac{\sigma}{\sigma - \mu^f} c_j$. We now write firm f's profit as a function of $(\mu^g)_{g \in \mathcal{F}}$:

$$\Pi^f = \frac{\left(\sum_{j \in f} (p_j - c_j)(-h_j'(p_j)) \right) \beta \left(\sum_{k \in f} h_k(p_k) \right)^{\beta - 1}}{\sum_{g \in \mathcal{F}} \left(\sum_{i \in g} h_i(p_i) \right)^\beta + (H^0)^\beta},$$

[23]The functions Φ and Ψ were introduced in Section VII.

$$
\begin{aligned}
&= \frac{\left(\sum_{j\in f}(\sigma-1)\frac{p_j-c_j}{p_j}h_j(p_j))\right)\beta\left(\sum_{k\in f}h_k(p_k)\right)^{\beta-1}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g}h_i(p_i)\right)^{\beta}+(H^0)^{\beta}},\\
&= \beta\frac{\sigma-1}{\sigma}\frac{\mu^f\left(\sum_{k\in f}h_k(p_k)\right)^{\beta}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g}h_i(p_i)\right)^{\beta}+(H^0)^{\beta}},\\
&= \beta\frac{\sigma-1}{\sigma}\frac{\mu^f\left(\left(\frac{\sigma}{\sigma-\mu^f}\right)^{1-\sigma}\sum_{k\in f}a_k c_k^{1-\sigma}\right)^{\beta}}{\sum_{g\in\mathcal{F}}\left(\left(\frac{\sigma}{\sigma-\mu^g}\right)^{1-\sigma}\sum_{k\in g}a_k c_k^{1-\sigma}\right)^{\beta}+(H^0)^{\beta}},\\
&= \beta(\sigma-1)\frac{\mu^f}{\sigma}\frac{\left(\frac{1}{1-\frac{\mu^f}{\sigma}}\right)^{\beta(1-\sigma)}T^f}{\sum_{g\in\mathcal{F}}\left(\frac{1}{1-\frac{\mu^g}{\sigma}}\right)^{\beta(1-\sigma)}T^g+(H^0)^{\beta}}, \text{ where } T^g=\left(\sum_{k\in g}a_k c_k^{1-\sigma}\right)^{\beta},\\
&= (\sigma'-1)\frac{\mu^f}{\sigma}\frac{\left(\frac{1}{1-\frac{\mu^f}{\sigma}}\right)^{1-\sigma'}T^f}{\sum_{g\in\mathcal{F}}\left(\frac{1}{1-\frac{\mu^g}{\sigma}}\right)^{1-\sigma'}T^g+(H^0)^{\beta}}, \text{ where } \sigma'=1+\beta(\sigma-1),\\
&= (\sigma'-1)\frac{\mu^{f\prime}}{\sigma'}\frac{\left(\frac{1}{1-\frac{\mu^{f\prime}}{\sigma'}}\right)^{1-\sigma'}T^f}{\sum_{g\in\mathcal{F}}\left(\frac{1}{1-\frac{\mu^{g\prime}}{\sigma'}}\right)^{1-\sigma'}T^g+(H^0)^{\beta}}, \text{ where } \mu^{g\prime}=\frac{\sigma'}{\sigma}\mu^g,\\
&= \frac{\sigma'-1}{\sigma'}\mu^{f\prime}\frac{\left(\frac{\sigma'}{\sigma'-\mu^{f\prime}}\right)^{1-\sigma'}T^f}{\sum_{g\in\mathcal{F}}\left(\frac{\sigma'}{\sigma'-\mu^{g\prime}}\right)^{1-\sigma'}T^g+H^{0\prime}}, \text{ where } H^{0\prime}=(H^0)^{\beta},
\end{aligned}
$$

which is the profit function that obtains in an auxiliary multiproduct-firm pricing game with CES demand, in which the elasticity of substitution is σ', the profile of types is $(T^g)_{g\in\mathcal{F}}$, and the value of the outside option is $H^{0\prime}$. It follows that $(\mu^{g*})_{g\in\mathcal{F}}$ is an equilibrium profile of ι-markups of the original game if and only if $\left(\frac{\sigma'}{\sigma}\mu^{g*}\right)_{g\in\mathcal{F}}$ is an equilibrium profile of ι-markups in the auxiliary game. Moreover, equilibrium profits in the original game are equal to equilibrium profits in the auxiliary game.

Note that firm f's market share (in value) in the original game given the profile of ι-markups $(\mu^g)_{g\in\mathcal{F}}$ is equal to that firm's market share in the auxiliary game given the profile of ι-markups $\left(\frac{\sigma'}{\sigma}\mu^g\right)_{g\in\mathcal{F}}$, since

$$
s^f=\frac{1}{\beta(\sigma-1)}\frac{\left(\sum_{j\in f}p_j h_j'(p_j)\right)\beta\left(\sum_{j\in f}h_j(p_j)\right)^{\beta-1}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g}h_i(p_i)\right)^{\beta}+(H^0)^{\beta}},
$$

$$
\begin{aligned}
&= \frac{\left(\sum_{j\in f} h_j(p_j)\right)^{\beta}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g} h_i(p_i)\right)^{\beta} + (H^0)^{\beta}}, \\
&= \frac{\left(\sum_{j\in f} h_j(p_j)\right)^{\beta}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g} h_i(p_i)\right)^{\beta} + (H^0)^{\beta}}, \\
&= \frac{\left(\frac{1}{1-\frac{\mu^f}{\sigma}}\right)^{\beta(1-\sigma)} T^f}{\sum_{g\in\mathcal{F}}\left(\frac{1}{1-\frac{\mu^g}{\sigma}}\right)^{\beta(1-\sigma)} T^g + (H^0)^{\beta}}, \\
&= \frac{\left(\frac{1}{1-\frac{\mu^{f\prime}}{\sigma'}}\right)^{1-\sigma'} T^f}{\sum_{g\in\mathcal{F}}\left(\frac{1}{1-\frac{\mu^{g\prime}}{\sigma'}}\right)^{1-\sigma'} T^g + (H^{0\prime})}.
\end{aligned}
$$

This implies that $(s^g)_{g\in\mathcal{F}}$ is an equilibrium profile of market shares in the original game if and only if it is an equilibrium profile of market shares in the auxiliary game.

Similarly, consumer surplus in the original game given the profile of ι-markups $(\mu^g)_{g\in\mathcal{F}}$ is equal to consumer surplus in the auxiliary game given the profile of ι-markups $\left(\frac{\sigma'}{\sigma}\mu^g\right)_{g\in\mathcal{F}}$:

$$
\begin{aligned}
CS &= \log\left(\sum_{g\in\mathcal{F}} T^g \left(\frac{\sigma}{\sigma-\mu^g}\right)^{\beta(1-\sigma)} + (H^0)^{\beta}\right), \\
&= \log\left(\sum_{g\in\mathcal{F}} T^g \left(\frac{\sigma'}{\sigma'-\mu^{g\prime}}\right)^{1-\sigma'} + H^{0\prime}\right).
\end{aligned}
$$

Hence, equilibrium consumer surplus in the original game is equal to consumer surplus in the auxiliary game. It follows that equilibrium social welfare in the original game is also equal to equilibrium social welfare in the auxiliary game.

(Nested) MNL demand. Under nested MNL demand, $\iota_j(p_j) = p_j/\lambda$ for every $j \in \mathcal{N}$. Hence, $r_j(\mu^f) = \lambda\mu^f + c_j$ for every j. Firm f's profit is given by:

$$
\begin{aligned}
\Pi^f &= \frac{\left(\sum_{j\in f}(p_j - c_j)(-h_j'(p_j))\right)\beta\left(\sum_{k\in f} h_k(p_k)\right)^{\beta-1}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g} h_i(p_i)\right)^{\beta} + (H^0)^{\beta}}, \\
&= \mu^f \frac{\beta\left(\sum_{k\in f} h_k(p_k)\right)^{\beta}}{\sum_{g\in\mathcal{F}}\left(\sum_{i\in g} h_i(p_i)\right)^{\beta} + (H^0)^{\beta}},
\end{aligned}
$$

$$= \mu^f \frac{\beta \left(\sum_{k\in f} \mathrm{e}^{\frac{a_k - c_k}{\lambda}}\right)^\beta \mathrm{e}^{-\beta\mu^f}}{\sum_{g\in\mathcal{F}} \left(\sum_{i\in g} \mathrm{e}^{\frac{a_i - c_i}{\lambda}}\right)^\beta \mathrm{e}^{-\beta\mu^g} + (H^0)^\beta},$$

$$= \mu^{f\prime} \frac{T^f \mathrm{e}^{-\mu^{f\prime}}}{\sum_{g\in\mathcal{F}} T^g \mathrm{e}^{-\mu^{g\prime}} + H^{0\prime}},$$

where $T^g = \left(\sum_{i\in g} \mathrm{e}^{\frac{a_i - c_i}{\lambda}}\right)^\beta$ and $\mu^{g\prime} = \beta\mu^g$ for every $g \in \mathcal{F}$, and $H^{0\prime} = (H^0)^\beta$. Hence, the original game is formally equivalent to an auxiliary pricing with MNL demand, in which the price sensitivity parameter is equal to 1, the profile of types is $(T^g)_{g\in\mathcal{F}}$, and the value of the outside option is $H^{0\prime}$. It is then straightforward to check that equilibrium market shares, consumer surplus and social welfare in the original game are the same as in the auxiliary game.

XIII.2 Algorithm

Numerically solving for the equilibrium of a multiproduct-firm pricing game in an industry with many firms and products can be a daunting task with standard methods, due to the high dimensionality of the problem. Exploiting the aggregative structure of the pricing game allows us to reduce this dimensionality tremendously: Instead of solving a system of $|\mathcal{N}|$ nonlinear equations in $|\mathcal{N}|$ unknowns, we only need to look for an $H > 0$ such that $\Gamma(H) = H$, where Γ is the aggregate fitting-in function. Of course, there usually will not be a closed-form expression for $\Gamma(\cdot)$, so we still need to compute this function numerically. But $\Gamma(H)$ is simple to compute as well, since all we need to do is solve for $|\mathcal{F}|$ separate equations, each with one unknown. Below, we describe how this general approach can be implemented to solve a multiproduct-firm pricing game with CES or MNL demands. Thanks to the formal equivalence results derived in Section XIII.1, this algorithm can also be used for pricing games with nested CES or MNL demand.

The algorithm uses two nested loops. The inner loop computes $\Omega(H) = \Gamma(H)/H$ for a given H. The outer loop iterates on H. We start by describing the inner loop. Fix some $H > 0$. We first need to compute $\mu^f = m^f(T^f/H)$ for every f. We have shown that μ^f solves equation (7) in the CES case, and equation (8) in the MNL case. These equations can be rewritten as follows:

$$0 = \psi^f(\mu^f) \equiv \begin{cases} \mu^f \left(1 - \frac{\sigma-1}{\sigma}\frac{T^f}{H}\left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma-1}\right) - 1 & \text{(CES)}, \\ \mu^f \left(1 - \frac{T^f}{H}\mathrm{e}^{-\mu^f}\right) - 1 & \text{(MNL)}. \end{cases} \tag{xl}$$

We solve equation (xl) numerically using the Newton-Raphson method with analytical derivatives. The usual problem with the Newton-Raphson method is that it may fail to converge if starting values are not good enough. This is potentially a major issue, because the value of $\Omega(H)$ used by the outer loop would then be incorrect. The following starting values guarantee convergence:

$$\mu_0^f = \begin{cases} \max\left(1, \sigma\left(1 - \left(\frac{H}{T^f}\right)^{\frac{1}{\sigma-1}}\right)\right) & \text{(CES)}, \\ \max\left(1, \log\frac{T^f}{H}\right) & \text{(MNL)}. \end{cases}$$

In fact, the Newton-Raphson method converges extremely fast (usually less than 5 steps). Notice, in addition, that this method can easily be vectorized by stacking up the μ^fs in a vector. Having computed μ^f for every firm f, we can calculate $\Omega(H)$ (see equation (15)).

The outer loop iterates on H to solve equation $\Omega(H) - 1 = 0$. This can be done by using standard derivative-based methods. The Jacobian can be computed analytically:

$$\Omega'(H) = -\frac{1}{H}\left(\frac{H^0}{H} + \sum_{f\in\mathcal{F}} \frac{T^f}{H} S'\left(\frac{T^f}{H}\right)\right),$$

where[24]

$$\frac{T^f}{H} S'\left(\frac{T^f}{H}\right) = \begin{cases} \frac{\mu^f - 1}{\frac{\sigma-1}{\sigma}\mu^f\left(1+(\sigma-1)(\mu^f-1)\frac{\mu^f}{\sigma-\mu^f}\right)} & \text{(CES)}, \\ \frac{\mu^f-1}{\mu^f\left(1+\mu^f(\mu^f-1)\right)} & \text{(MNL)}. \end{cases}$$

We use the value of H that would prevail under monopolistic competition as starting value ($H^{ini} = H^0 + \sum_{f\in\mathcal{F}} T^f \left(1 - \frac{1}{\sigma}\right)^{\sigma-1}$ under CES demand, $H^{ini} = H^0 + \sum_{f\in\mathcal{F}} T^f \mathrm{e}^{-1}$ under MNL demand), and we always obtain convergence (usually in about 10 steps).[25]

XIII.3 Formulas for m' and S' and Preliminary Lemmas

Applying the implicit function theorem to equations (7) and (8) yields:

$$\text{(CES)} \quad m'(x) = \frac{\frac{\sigma-1}{\sigma} m(x)^2 \left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-1}}{1 + \left(\frac{\sigma-1}{\sigma}\right)^2 m(x)^2 x \left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-2}}, \tag{xli}$$

$$\text{(MNL)} \quad m'(x) = \frac{m(x)^2 \, \mathrm{e}^{-m(x)}}{1 + m(x)^2 x \, \mathrm{e}^{-m(x)}}. \tag{xlii}$$

Let $\alpha = (\sigma - 1)/\sigma$ in the CES case and $\alpha = 1$ in the MNL case. Note that $m = \sigma/(\sigma - (\sigma - 1)S)$ in the CES case, and $m = 1/(1 - S)$ in the MNL case. Therefore, in both cases, $m = 1/(1 - \alpha S)$, $S = \frac{1}{\alpha}\frac{m-1}{m}$, and $S' = \frac{m'}{\alpha m^2}$. This implies in particular that

$$\text{(CES)} \quad \frac{1}{\alpha}\frac{m(x) - 1}{m(x)} = S(x) = x\left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-1},$$

[24] We derive these formulas in Section XIII.3.

[25] In Breinlich, Nocke, and Schutz (2015), we use this algorithm to calibrate an international trade model with two countries, 160 manufacturing industries, CES demand and oligopolistic competition.

$$\text{(MNL)} \quad \frac{m(x)-1}{m(x)} = S(x) = x\,\mathrm{e}^{-m(x)}\,.$$

This allows us to obtain expressions for $S'(x)$, which do not explicitly depend on the terms $(1-m(x)/\sigma)^{\sigma-1}$, $(1-m(x)/\sigma)^{\sigma-2}$ and $\mathrm{e}^{-m(x)}$:

$$\text{(CES)} \quad xS'(x) = \frac{m(x)-1}{\frac{\sigma-1}{\sigma}m(x)\left(1+\frac{\sigma-1}{\sigma}\frac{m(x)}{1-m(x)/\sigma}(m(x)-1)\right)}, \tag{xliii}$$

$$\text{(MNL)} \quad xS'(x) = \frac{m(x)-1}{m(x)\,(1+m(x)(m(x)-1))}. \tag{xliv}$$

Formulas (xliii) and (xliv) are used at the end of Section XIII.2.

Next, we use the fact that $m = 1/(1-\alpha S)$ to replace $m(x)$ in the right-hand side of equations (xliii) and (xliv). In the MNL case, we have that:

$$xS'(x) = \frac{S(x)}{1+m^2(x)S(x)} = \frac{S(x)}{1+\frac{S(x)}{(1-S(x))^2}} = \frac{S(x)(1-S(x))^2}{1-S(x)+S(x)^2}.$$

In the CES case, we have that:

$$\begin{aligned} xS'(x) &= \frac{S(x)}{1+\alpha^2 m^2(x)\frac{S(x)}{1-m(x)/\sigma}}, \\ &= \frac{S(x)}{1+\alpha^2\frac{1}{(1-\alpha S(x))^2}\frac{S(x)(1-\alpha S(x))}{1-S(x)}}, \\ &= \frac{S(x)}{1+\alpha\frac{S(x)}{(1-S(x))(1-\alpha S(x))}}, \\ &= \frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S^2(x)}. \end{aligned}$$

Therefore, in both cases:

$$xS' = \frac{S(x)}{1+\alpha\frac{S(x)}{(1-S(x))(1-\alpha S(x))}}, \tag{xlv}$$

$$= \frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S^2(x)}. \tag{xlvi}$$

Let $\varepsilon(x) = xS'(x)/S(x)$ be the elasticity of S. We prove the following technical lemmas:

Lemma XXV. $\varepsilon' < 0$.

Proof. Using equation (xlv), we see that

$$\varepsilon(x) = \frac{1}{1+\alpha\frac{S(x)}{(1-S(x))(1-\alpha S(x))}}.$$

Since $S' > 0$, it follows that $\varepsilon' < 0$. □

Lemma XXVI. *$S'' < 0$. Therefore, S is strictly subadditive.*

Proof. Using equation (xlv) and the fact that $S(x) = x(1-m(x)/\sigma)^{\sigma-1}$ in the CES case and $m(x) = x\exp(-m(x))$ in the MNL case, we see that

$$\text{(CES)} \quad S'(x) = \frac{\left(1-\frac{m(x)}{\sigma}\right)^{\sigma-1}}{1+\alpha\frac{S(x)}{(1-S(x))(1-\alpha S(x))}},$$
$$\text{(MNL)} \quad S'(x) = \frac{\mathrm{e}^{-m(x)}}{1+m(x)^2 S(x)}.$$

Since $m' > 0$ and $S' > 0$, it follows that $S'' < 0$.

Let $y > 0$, and define $\xi : x \in \mathbb{R}_{++} \mapsto S(x+y) - S(x) - S(y)$. Note that $\lim_{x\to 0}\xi(x) = 0$, and that

$$\xi'(x) = S'(x+y) - S'(x) < 0,$$

since $S'' < 0$. Therefore, ξ is strictly decreasing, and $\xi < 0$. □

XIII.4 Proof of Proposition 6

Proof. The fact that $m' > 0$, $S' > 0$, and $\pi'(= m') > 0$ can be seen by inspecting equation (xli), (xlii), and (xlv).

Applying the implicit function theorem to equation $\Omega(H) = 1$ yields:

$$\frac{dH^*}{dT^f} = \frac{S'\left(\frac{T^f}{H^*}\right)}{\frac{H^0}{H^*} + \sum_{g\in\mathcal{F}} \frac{T^g}{H^*} S'\left(\frac{T^g}{H^*}\right)} > 0. \tag{xlvii}$$

Hence, equilibrium consumer surplus is increasing in types.

Note that

$$\frac{d\left(\frac{T^f}{H^*}\right)}{dT^f} = \frac{1}{H^*}\left(1 - \frac{T^f}{H^*}\frac{dH^*}{dT^f}\right) = \frac{1}{H^*}\left(1 - \frac{\frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right)}{\frac{H^0}{H^*} + \sum_{g\in\mathcal{F}} \frac{T^g}{H^*} S'\left(\frac{T^g}{H^*}\right)}\right) > 0,$$

and that, for $g \neq f$,

$$\frac{d\left(\frac{T^g}{H^*}\right)}{dT^f} = -\frac{T^g}{H^{*2}}\frac{dH^*}{dT^f} < 0.$$

Applying the chain rule allows us to conclude that firm f's equilibrium markup, market share and profit are increasing in T^f and decreasing in T^g ($g \neq f$).

Next, we turn our attention to social welfare. Let $x^g = T^g/H^*$ for every g and $x^0 = H^0/H^*$. Social welfare is given by

$$W^* = \log H^* + \sum_{g \in \mathcal{F}} (m(x^g) - 1).$$

Therefore,

$$\begin{aligned}
\frac{dW^*}{dT^f} &= \frac{1}{H^*}\left(\frac{dH^*}{dT^f}\left(1 - \sum_{g\in\mathcal{F}} x^g m'(x^g)\right) + m'(x^f)\right), \\
&= \frac{1}{H^*}\left(\frac{S'(x^f)}{x^0 + \sum_{g\in\mathcal{F}} x^g S'(x^g)}\left(1 - \sum_{g\in\mathcal{F}} x^g \alpha \frac{S'(x^g)}{(1-\alpha S(x^g))^2}\right) + \alpha \frac{S'(x^f)}{(1-\alpha S(x^f))^2}\right), \\
&\geq \frac{S'(x^f)}{H^*\left(x^0 + \sum_{g\in\mathcal{F}} x^g S'(x^g)\right)}\left(1 + \alpha \sum_{g\in\mathcal{F}} x^g S'(x^g)\left(\frac{1}{(1-\alpha S(x^f))^2} - \frac{1}{(1-\alpha S(x^g))^2}\right)\right), \\
&= \frac{S'(x^f)}{H^*\left(x^0 + \sum_{g\in\mathcal{F}} x^g S'(x^g)\right)}\left(1 + \alpha \sum_{g\in\mathcal{F}} \frac{s^g(1-s^g)(1-\alpha s^g)}{1 - s^g + \alpha (s^g)^2}\left(\frac{1}{(1-\alpha s^f)^2} - \frac{1}{(1-\alpha s^g)^2}\right)\right), \\
&> \frac{S'(x^f)}{H^*\left(x^0 + \sum_{g\in\mathcal{F}} x^g S'(x^g)\right)}\left(1 + \sum_{g\in\mathcal{F}} \underbrace{\alpha \frac{s^g(1-s^g)(1-\alpha s^g)}{1 - s^g + \alpha (s^g)^2}\left(1 - \frac{1}{(1-\alpha s^g)^2}\right)}_{\equiv \psi_\alpha(s^g)}\right),
\end{aligned}$$

where the second line follows from equation (xlvii) and the fact that $m = \frac{1}{1-\alpha S}$, and the fourth line follows from equation (xlvi).

If we can show that $1 + \sum_{i=1}^n \psi_\alpha(s_i) \geq 0$ for every $\alpha \in (0,1]$, $n \geq 2$, and $(s_i)_{1\leq i\leq n} \in [0,1)^n$ such that $\sum_{i=1}^n s_i \leq 1$, then we are done. Routine calculations show that $\psi_\alpha(s) \geq \psi_1(s) \equiv \psi(s)$ for every s. Therefore, all we need to do is show that $1 + \sum_{i=1}^n \psi(s_i) \geq 0$ for every $n \geq 2$ and $(s_i)_{1\leq i\leq n} \in [0,1)^n$ such that $\sum_{i=1}^n s_i \leq 1$. Note that $\psi(s) = s^2(s-2)/(1-s+s^2)$. Routine calculations show that:

(i) ψ is concave on $[0, 1/2]$.

(ii) $\psi(0) = 0$.

(iii) $\psi(s) + \psi(1-s) = -1$ for every $s \in [0,1]$.

(iv) $\psi(s) > -s$ (resp. $\psi(s) < -s$) if and only if $s < 1/2$ (resp. $s > 1/2$).

(v) ψ is decreasing.

By point (iv), if $s_i \leq 1/2$ for every i, then $1+\sum_{i=1}^{n} \psi(s_i) \geq 0$. Next, let $(s_i)_{1\leq i\leq n}$ such that $s_i > 1/2$ for some i. Assume without loss of generality that $s_n > 1/2$. Then, $\sum_{i=1}^{n-1} s_i < 1/2$. We claim that

$$\sum_{i=1}^{n-1} \psi(s_i) \geq \psi\left(\sum_{i=1}^{n-1} s_i\right). \tag{xlviii}$$

To see this, let $x, y \in [0, 1/2]$ such that $x + y \leq 1/2$, and define

$$\xi : t \in [0, y] \mapsto \psi(x+t) - \psi(x) - \psi(t).$$

By point (ii), $\xi(0) = 0$. By point (i), $\xi' \leq 0$. Therefore, $\xi(t) \leq 0$ for every t. In particular, $\psi(x+y) \leq \psi(x) + \psi(y)$. Property (xlviii) follows by induction on n. Therefore,

$$1 + \sum_{i=1}^{n} \psi(s_i) \geq 1 + \psi\left(\sum_{i=1}^{n-1} s_i\right) + \psi(s_n) \geq 1 + \psi(1 - s_n) + \psi(s_n) = 0,$$

where the second inequality follows by point (v), and the last equality follows by point (iii). □

XIV Comparative Statics

XIV.1 Proof of Proposition 3

Proof. The first part of the proposition follows immediately from equation (ii), Theorem 1 and Lemma I.

Next, we prove that largest and smallest (in terms of the value of H) equilibria exist. If there is a finite number of equilibrium aggregators, then this is trivial. Next, assume that there is an infinite number of equilibria. We have shown in the proof of Lemma J that $\Omega(H) > 1$ for H low enough and $\Omega(H) < 1$ for H high enough. Therefore, the set of equilibrium aggregators is contained in a closed interval $[\underline{H}, \overline{H}]$, with $\underline{H} > 0$. Put

$$\overline{H}^* \equiv \sup\left\{H \in [\underline{H}, \overline{H}] : \quad \Omega(H) = 1\right\}.$$

Let $(H^n)_{n\geq 0}$ be a sequence such that $\Omega(H^n) = 1$ for all n and $H^n \underset{n\to\infty}{\longrightarrow} \overline{H}^*$. Since Ω is continuous on $[\underline{H}, \overline{H}]$, we can take limits and obtain that $\Omega(\overline{H}^*) = 1$. Therefore,

$$\overline{H}^* = \max\left\{H \in [\underline{H}, \overline{H}] : \quad \Omega(H) = 1\right\}$$

is the highest equilibrium aggregator level. The existence of a lowest equilibrium aggregator follows from the same line of argument. □

XIV.2 Proof of Proposition 4

Proof. Given the outside option $H^0 \geq 0$, $H > 0$ is an equilibrium aggregator level if and only if $\Omega(H, H^0) = 1$, where

$$\Omega(H, H^0) = \frac{H^0 + \sum_{f \in \mathcal{F}} \sum_{j \in f} h_j \left(r_j \left(m^f(H)\right)\right)}{H}.$$

Let $H^{0\prime} > H^0 \geq 0$, and note that $\Omega(H, H^{0\prime}) > \Omega(H, H^0)$ for all $H > 0$. Let $\overline{H}$ and $\underline{H}$ (resp. $\overline{H}'$ and $\underline{H}'$) be the highest and lowest equilibrium aggregator levels when the outside option is H^0 (resp. $H^{0\prime}$). We know from the proof of Lemma J that $\Omega(H, H^0) \geq 1$ for all $H \leq \underline{H}$. Therefore, for all $H \leq \underline{H}$,

$$\Omega(H, H^{0\prime}) > \Omega(H, H^0) \geq 1.$$

It follows that, when the outside option is $H^{0\prime}$, there is no equilibrium aggregator level weakly below $\underline{H}$. Therefore, $\underline{H} < \underline{H}'$. The fact that $\overline{H} < \overline{H}'$ follows from the same line of argument. This establishes point (iii) in the proposition.

Points (i), (ii) and (iv) follow from the fact that a firm's profit is equal to its ι-markup minus one (Theorem 1), m^f is decreasing (Lemma I), and r_j is increasing (Lemma E).

The result on entry follows from the same line of argument: After entry takes place, Ω shifts upward. □

XIV.3 On the Impact of Production Costs on Equilibrium Consumer Surplus

The goal of this section is to construct a discrete/continuous choice model $((h_j)_{j \in \mathcal{N}}, H^0)$ and a firm partition $\mathcal{F}$ such that: (a) The pricing game $((h_j)_{j \in \mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ has a unique equilibrium for every $(c_j)_{j \in \mathcal{N}}$; (b) There exists a marginal cost vector $(c_j)_{j \in \mathcal{N}}$ and a product k such that, starting from $(c_j)_{j \in \mathcal{N}}$, a small increase in c_k *raises* the equilibrium aggregator level.

Fix an arbitrary pricing game $((h_j)_{j \in \mathcal{N}}, H^0, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$. We start by deriving a necessary and sufficient condition under which the aggregate fitting-in function shifts upward (locally) after an increase in c_j ($j \in f$).[26] In the following, we make explicit the dependence of the function m^f on c_j by writing $m^f(H, c_j)$. We also write $r_k(\mu^f, c_k)$ for every k. Differentiating equation (14) with respect to c_j and μ^f, and using equation (14) to eliminate H, we obtain:

$$\frac{\partial m^f}{\partial c_j} = -\frac{m^f(m^f - 1)(-\gamma_j')\frac{\partial r_j}{\partial c_j}}{\sum_{k \in f}\left(\gamma_k + m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-\gamma_k')\right)}.$$

[26] To simplify the exposition, we assume that firm f sets finite prices for all its products. This condition holds in the example we construct below.

It is straightforward to check that $\partial r_j/\partial c_j > 0$. Therefore, $\partial m^f/\partial c_j < 0$.

Next, let $H^f(H, c_j) \equiv \sum_{k \in f} h_k(r_k(m^f(H, c_j), c_k))$ be firm f's contribution to the aggregator. Note that an infinitesimal increase in c_j implies a local upward shift in the aggregate fitting-in function if and only if $\partial H^f/\partial c_j > 0$. Let $\xi = \sum_{k \in f}\left(\gamma_k + m^f(m^f - 1)\frac{\partial r_k}{\partial \mu^f}(-\gamma'_k)\right)$, and, as in Section V.2.3, $\omega^f = (\mu^f - 1)/\mu^f$, and $\theta_k = h'_k/\gamma'_k$ for every k. Note that $\frac{\partial r_k}{\partial \mu^f} = \frac{\gamma_k}{(-\gamma'_k)\mu^f(1-\omega^f\theta_k)}$ (see Lemma E). Then,

$$
\begin{aligned}
\frac{\partial H^f}{\partial c_j} &= \frac{\partial r_j}{\partial c_j}h'_j + \frac{\partial m^f}{\partial c_j}\sum_{k \in f}\frac{\partial r_k}{\partial \mu^f}h'_k, \\
&= \frac{1}{\xi}\frac{\partial r_j}{\partial c_j}\left(-(-h'_j)\xi + m^f(m^f-1)(-\gamma'_j)\sum_{k \in f}\frac{\partial r_k}{\partial \mu^f}(-h'_k)\right), \\
&= \frac{1}{\xi}\frac{\partial r_j}{\partial c_j}\sum_{k \in f}\left(-(-h'_j)\left(\gamma_k + m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-\gamma'_k)\right) + (-\gamma'_j)m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-h'_k)\right), \\
&= \frac{-\gamma'_j}{\xi}\frac{\partial r_j}{\partial c_j}\sum_{k \in f}\gamma_k\left(-\theta_j\left(1 + \frac{m^f-1}{1-\omega^f\theta_k}\right) + \frac{(m^f-1)\theta_k}{1-\omega^f\theta_k}\right), \\
&= \frac{-\gamma'_j}{\xi}\frac{\partial r_j}{\partial c_j}\sum_{k \in f}\gamma_k\left(-\theta_j + \frac{\omega^f}{1-\omega^f}\frac{\theta_k-\theta_j}{1-\omega^f\theta_k}\right).
\end{aligned}
$$

If $f = \{1, 2\}$ and $j = 1$, then $\partial H^f/\partial c_1 > 0$ if and only if

$$-\gamma_1\theta_1 + \gamma_2\left(-\theta_1 + \frac{\omega^f}{1-\omega^f}\frac{\theta_2-\theta_1}{1-\omega^f\theta_2}\right) > 0, \qquad \text{(xlix)}$$

where $\omega^f = \frac{m^f(H,c_1)-1}{m^f(H,c_1)}$, the functions γ_1 and θ_1 are evaluated at price $p_1 = r_1(m^f(H, c_1), c_1)$, and the functions γ_2 and θ_2 are evaluated at price $p_2 = r_2(m^f(H, c_1), c_2)$.

The next step is to find a product pair $(h_1, h_2) \in (\mathcal{H}^\iota)^2$, a marginal cost pair (c_1, c_2), and an aggregator level $H^* > 0$ such that firm f satisfies condition (b) in Theorem II, and condition (xlix) holds. Let product h_2 be a CES product with quality a_2 and $\sigma = 2$: $h(p_2) = a_2/p_2$. Let $h_1(p_1) = 1/\log(1 + \mathrm{e}^{p_1})$. Routine calculations show that $h_1 \in \mathcal{H}^\iota$, $\bar{\mu}_1 = \bar{\mu}_2 = 2$, $\lim_{p_1\to\infty} h_1(p_1) = 0$, and ρ_1 is strictly increasing. Therefore, firm $f = \{1, 2\}$ satisfies condition (b) in Theorem II. Moreover, using the properties of CES products ($\theta_2 = 2$) allows us to simplify condition (xlix) as follows:

$$-\gamma_1\theta_1 + \gamma_2\left(-\theta_1 + \frac{\omega^f}{1-\omega^f}\frac{2-\theta_1}{1-2\omega^f}\right) > 0, \qquad \text{(l)}$$

Fix $c_2 > 0$ at some arbitrary value. We need to find $H^* > 0$, $a_2 > 0$ and $c_1 > 0$ such that condition (xlix) holds.

Let $\mu^f \in (1,2)$ and $\omega^f = (\mu^f - 1)/\mu^f$. Note that, as c_1 tends to zero, $r_1(\mu^f, c_1)$ converges to a strictly positive real $p_1 = r_1(0, \mu^f)$, which is the unique solution of equation $\iota_1(p_1) = \mu^f$, or, equivalently, $\chi_1(p_1) = \omega^f$. At the limit, the term in parentheses in equation (l) can then be rewritten as follows:

$$\psi(p_1) = -\theta_1(p_1) + \frac{\chi_1(p_1)}{1-\chi_1(p_1)} \frac{2-\theta_1(p_1)}{1-2\chi_1(p_1)}.$$

Studying the function ψ, we show that $\psi(p_1) > 0$ (and $\iota_1(p_1) > 1$) for p_1 high enough. Fix such a p_1, and let $\mu^f \equiv \iota_1(p_1)$ and $\omega^f = (\mu^f - 1)/\mu^f$. Then, by definition of p_1,

$$-\theta_1(r_1(\mu^f, 0)) + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1(r_1(\mu^f, 0))}{1-2\omega^f} > 0.$$

Therefore, by continuity,

$$-\theta_1(r_1(\mu^f, c_1)) + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1(r_1(\mu^f, c_1))}{1-2\omega^f} > 0$$

for $c_1 > 0$ small enough. Fix such a c_1.

Let us now inspect the expression in the left-hand side of condition (l) (recall that, since good 2 is a CES product with $\sigma = 2$, $\gamma_2 = h_2/2$):

$$-\gamma_1(r_1(\mu^f, c_1))\theta_1(r_1(\mu^f, c_1)) + \frac{1}{2}\frac{a_2}{r_2(\mu^f, c_2)} \left(-\theta_1(r_1(\mu^f, c_1)) + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1(r_1(\mu^f, c_1))}{1-2\omega^f} \right).$$

Clearly, the above expression is strictly positive for high enough a_2. Fix such an a_2. Recall that $m^f(\cdot, c_1)$ is continuous, and decreases from $\bar{\mu}^f (= 2)$ to 1 as H increases from 0 to ∞ (Lemma I). Therefore, there exists $H^* > 0$ such that $m^f(H^*, c_1) = \mu^f$. This concludes the second step of our construction.

The last step is to construct a second firm, firm g, such that the pricing game between firms f and g gives rise to a unique equilibrium, and the equilibrium aggregator level is H^*. Before constructing firm g, we state and prove the following lemma:

Lemma XXVII. *Let $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$, and $(h_j)_{j\in\mathcal{N}} \in (\mathcal{H}^\iota)^{\mathcal{N}}$ such that $\bar{\mu}_j = \bar{\mu} < \infty$, $\lim_{p_j\to\infty} h_j(p_j) = 0$, and ρ_j is non-decreasing for every $j \in \mathcal{N}$. Suppose that a monopolist owns all the products in $\mathcal{N}$, and that consumers have access to an outside option $H^0 > 0$. Then, the monopolist's profit-maximization problem has a unique solution. The aggregator level at the monopolist's optimum, $\hat{H}(H^0)$, is a strictly increasing function of H^0. Moreover, $\lim_{H^0\to 0} \hat{H}(H^0) = 0$, and $\lim_{H^0\to\infty} \hat{H}(H^0) = \infty$.*

Proof. We know from Lemma H that the monopoly problem has a unique solution for every $H^0 > 0$. Therefore, the function $\hat{H}(\cdot)$ is well defined. The monopolist's optimal ι-markup, denoted $\hat{\mu}(H^0) \in (1, \bar{\mu}^f)$, is the unique solution of equation (12). It is straightforward to show, e.g., by applying the implicit function theorem to equation (12), that $\hat{\mu}$ is continuous and strictly decreasing. It follows that

$$\hat{H}(H^0) = H^0 + \sum_{j \in \mathcal{N}} h_j(r_j(\hat{\mu}(H^0)))$$

is strictly increasing in H^0. The monopolist earns $\hat{\mu}(H^0) - 1$ at its optimum. Let $m(\cdot)$ be the monopolist's fitting-in function. Then, by definition of m, $m(\hat{H}(H^0)) = \hat{\mu}(H^0)$.

Clearly, $\lim_{H^0 \to \infty} \hat{H}(H^0) = \infty$. By monotonicity, $\underline{H} = \lim_{H^0 \to 0} \hat{H}(H^0)$ exists, and is non-negative. Assume for a contradiction that $\underline{H} > 0$. Then, for every $H^0 > 0$,

$$\hat{\mu}(H^0) = m(\hat{H}(H^0)) < m(\underline{H}) < \bar{\mu}.$$

For every $\mu \in (1, \bar{\mu})$ and $H^0 > 0$, let $\pi(\mu, H^0)$ be the monopolist's profit when it sets the ι-markup μ, and the value of the outside option is H^0. Note that, for every $H^0 > 0$ and $\mu \in (1, \bar{\mu})$,

$$\pi(\mu, H^0) \leq \hat{\mu}(H^0) - 1 \leq m(\underline{H}) - 1.$$

Therefore,

$$\bar{\pi} \equiv \sup_{H^0 > 0,\ \mu \in (1, \bar{\mu})} \pi(\mu, H^0) \leq m(\underline{H}) - 1 < \bar{\mu} - 1.$$

Moreover, using the definition of the ι-markup μ and the function γ_j ($j \in \mathcal{N}$), we can rewrite $\pi(\mu, H^0)$ as follows:

$$\pi(\mu, H^0) = \mu \frac{\sum_{j \in \mathcal{N}} \gamma_j(r_j(\mu))}{H^0 + \sum_{j \in \mathcal{N}} h_j(r_j(\mu))}.$$

Note that, for every $\mu \in (1, \bar{\mu})$,

$$\bar{\pi} \geq \mu \frac{\sum_{j \in \mathcal{N}} \gamma_j(r_j(\mu))}{\sum_{j \in \mathcal{N}} h_j(r_j(\mu))} = \mu \frac{\sum_{j \in \mathcal{N}} \gamma_j(r_j(\mu))}{\sum_{j \in \mathcal{N}} \rho_j(r_j(\mu)) \gamma_j(r_j(\mu))} \geq \mu \frac{\bar{\mu} - 1}{\bar{\mu}} \frac{\sum_{j \in \mathcal{N}} \gamma_j(r_j(\mu))}{\sum_{j \in \mathcal{N}} \gamma_j(r_j(\mu))} = \mu \frac{\bar{\mu} - 1}{\bar{\mu}},$$

where the second inequality comes from the fact that, for every j, ρ_j is non-decreasing and $\lim_\infty \rho_j = \bar{\mu}/(\bar{\mu} - 1)$ by Lemma A-(f). Taking the limit as μ tends to $\bar{\mu}$ allows us to conclude that $\bar{\pi} \geq \bar{\mu} - 1$, which is a contradiction. □

Firm f satisfies all the assumptions in Lemma XXVII. Therefore, the function $\hat{H}(\cdot)$ is a bijection from $(0, \infty)$ to $(0, \infty)$, and there exists a unique $H^0 > 0$ such that $\hat{H}(H^0) = H^*$. By definition of $\hat{H}$, this means that

$$H^* = H^0 + \sum_{k \in f} h_k(r_k(m^f(H^*, c_1), c_k)).$$

Next, we construct a firm g such that, when the aggregator level is H^*, firm g's contribution to the aggregator is H^0. To do so, we rely on the results derived in Section 5. Let g be an arbitrary multiproduct firm selling only CES products (with a common σ). Denote firm g's type by $T^g > 0$. We know that, when the aggregator level is H^*, firm g's contribution to the aggregator is $S(T^g/H^*)H^*$. Moreover, $S(\cdot)$ is continuous and strictly increasing, and it is straightforward to show that $\lim_{x\to 0} S(x) = 0$ and $\lim_{x\to\infty} S(x) = 1$. Therefore, there exists a unique $\hat{T}^g$ such that $S(\hat{T}^g/H^*)H^* = H^0$.

We can conclude. We have constructed a multiproduct-firm pricing game with two firms, f and g. By construction, firm f satisfies condition (b) in Theorem II. Since firm g only sells CES products with a common σ, firm g satisfies condition (a) in Theorem II. Therefore, the pricing game between firms f and g has a unique equilibrium for every marginal cost vector for firm f and for every value of T^g. When firm f's marginal costs are equal to c_1 and c_2, as defined above, and firm g's type is $\hat{T}^g$, the equilibrium aggregator level is H^*. An infinitesimal increase in the value of c_1 induces a local upward shift in the aggregate fitting-in function. Since that function has a finite limit when $H \to \infty$ and has a unique fixed point, it follows that the equilibrium value of the aggregator increases. Therefore, consumer surplus increases, and both firms' profits decrease.

XIV.4 On the Impact of Production Costs on a Firm's Equilibrium Profit

The goal of this section is to construct a pricing game in which a firm's equilibrium profit is a non-monotonic function of that firm's marginal cost. We do so numerically.

We work with two single-product firms: $\mathcal{N} = \{1, 2\}$, and $\mathcal{F} = \{\{1\}, \{2\}\}$. Products are symmetric: $h_1(p) = h_2(p) = h(p)$. We use the following function:

$$h(p) = \exp\left(-\frac{1}{2}p^{\frac{1}{4}}\right).$$

Note that

$$\iota(p) = \frac{1}{8}\left(6 + p^{\frac{1}{4}}\right),$$

and that

$$\gamma(p) = \frac{p^{\frac{1}{4}}}{6 + p^{\frac{1}{4}}} \exp\left(-\frac{1}{2}p^{\frac{1}{4}}\right) = \frac{p^{\frac{1}{4}}}{6 + p^{\frac{1}{4}}} h(p) < h(p),$$

so $h \in \mathcal{H}^\iota$. Since $h \in \mathcal{H}^\iota$, it follows that the pricing game $((h_j)_{j\in\mathcal{N}}, 0, \mathcal{F}, (c_1, c_2))$ has an equilibrium for every (c_1, c_2).

Since the function

$$\rho(p) = \frac{h(p)}{\gamma(p)} = \frac{6 + p^{\frac{1}{4}}}{p^{\frac{1}{4}}}$$

is strictly decreasing, none of the uniqueness conditions derived in Section V applies. We will therefore need to establish equilibrium uniqueness manually.

In the following, we focus on the special case in which $c_2 = 0.01$ and $c_1 \in [5, 50]$. We first show numerically that the pricing equilibrium is unique for every $c_1 \in \{5, 10, 15, \ldots, 45, 50\}$. Note that, for every c_1, $H^{mc}(c_1)$, the monopolistic competition aggregator level (given c_1 and c_2) is an upper bound for the set of equilibrium aggregator levels. Moreover, $H^{mc}(c_1)$ is strictly decreasing in c_1. It follows that $H^{mc}(5)$ is an upper bound for the set of equilibrium aggregator levels for any $c_1 \geq 5$. Numerically, we find that $H^{mc}(c_1) \simeq 0.62$. We can therefore narrow down our search for equilibrium aggregator levels to the interval $(0, 0.62)$.

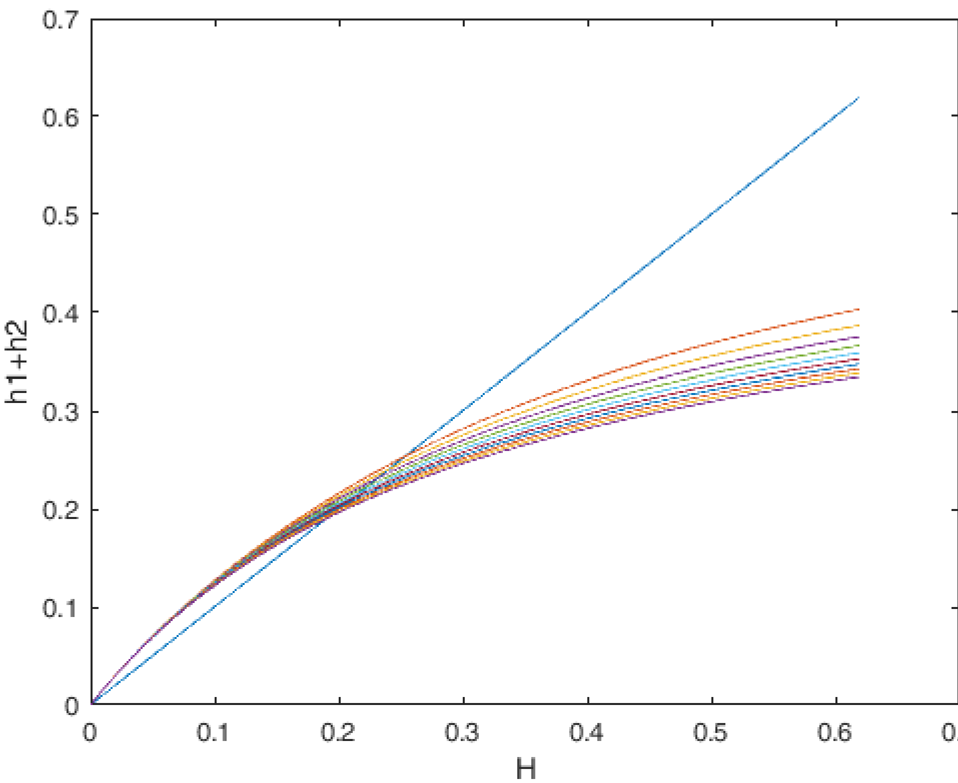

Figure 2: Aggregate Fitting-in Functions for $c_1 \in \{5, 10, 15, \ldots, 45, 50\}$

Figure 2 plots aggregate fitting-in functions for $c_1 \in \{5, 10, 15, \ldots, 45, 50\}$. The graph has been constructed with a step size of 0.001. The blue line is the 45-degree line. The curves represents aggregate fitting-in functions for different values of c_1. We can see that each curve intersects the 45-degree line only once on $(0, 0.62)$, which shows that the equilibrium is unique. (Since $\lim_{p \to \infty} h(p) = 0$, the aggregate fitting-in functions also intersect the 45-degree line at $H = 0$. Of course, $H = 0$ cannot be an equilibrium aggregator level.)

Next, we show that firm 1's equilibrium profit is non-monotonic in c_1. For every $c_1 \in \{5, 6, 7, \ldots, 49, 50\}$, we compute the equilibrium aggregator level and firm 1's equilibrium profit. Figure 3 depicts the relationship between firm 1's profit and c_1. That relationship is clearly non-monotonic. (Of course, we have not shown that the equilibrium is unique for every $c_1 \in \{5, 6, \ldots, 49, 50\} \setminus \{5, 10, \ldots, 45, 50\}$, but Figure 3 clearly shows that firm 1's profit is also non-monotonic on $\{5, 10, \ldots, 45, 50\}$.)

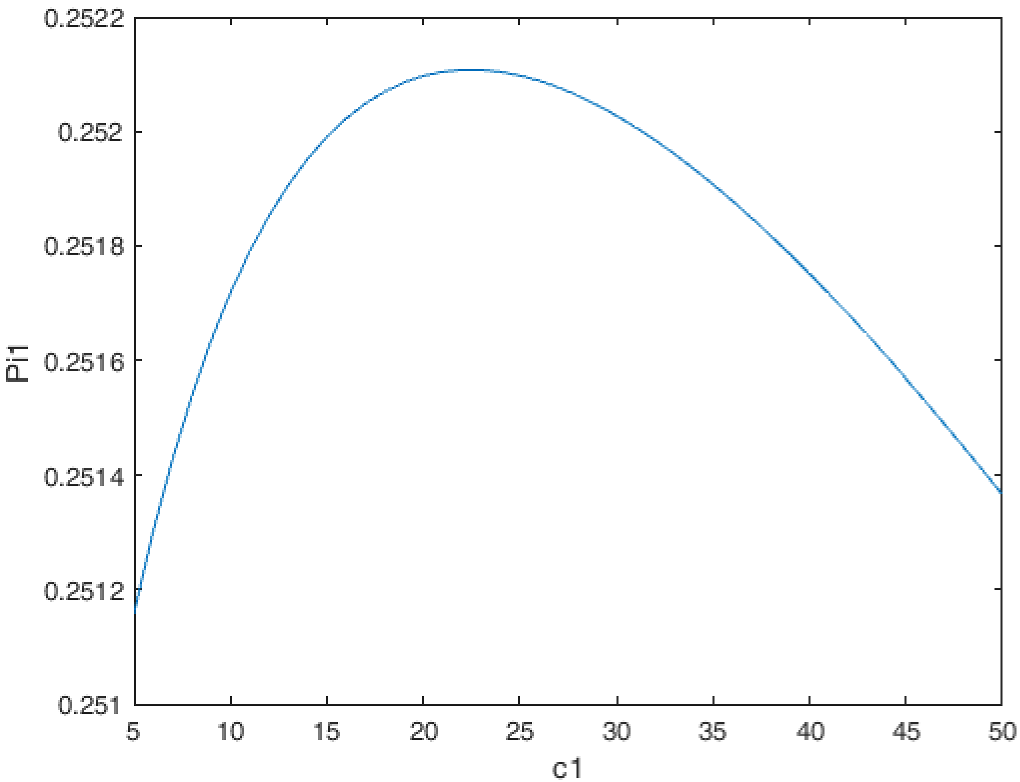

Figure 3: Equilibrium Profit of Firm 1

XV Application: Merger Policy

In this section, we apply the aggregative games approach to analyze static and dynamic merger policy. Throughout, we assume that demand is either of the (nested) CES or MNL forms so that the type aggregation property (see Section 5) holds.

Much of the existing literature on merger analysis, including Farrell and Shapiro (1990) and Nocke and Whinston (2010), relies on the homogeneous-goods Cournot model. In Section XV.1, we provide a necessary and sufficient condition for a merger to increase consumer surplus (resp. social welfare), and a sufficient condition for a merger to have a positive external effect, thereby extending the classic results of Farrell and Shapiro (1990) to the case of price competition between multiproduct firms. In Section XV.2, we extend Nocke and Whinston (2010) by showing that a myopic merger approval policy is dynamically optimal if the goal is to maximize discounted consumer surplus. Formal proofs are provided in Section XV.3.

For both our static and dynamic merger analysis, the type aggregation property turns out to be very useful. In principle, merger-specific synergies could take many forms: Some of the marginal costs of the merged firms' may go down (while those of others may go up); some of the products' qualities may improve (while others may degrade); the merged entity may offer new products (while possibly withdrawing others). The type aggregation property implies that we do not need to impose any restrictions as all relevant information can be summarized in the merged firm's post-merger type.

XV.1 Static Merger Analysis

Partition the set of firms, set $\mathcal{F}$, into $\mathcal{I}$ (the insiders) and $\mathcal{O}$ (the outsiders), and suppose that the insiders merge. Let H^* (resp., $\hat{H}^*$) denote the equilibrium value of the aggregator before (resp., after) the merger. As consumer surplus is increasing in the value of that aggregator, we say that the merger is *CS-increasing* (resp., *CS-decreasing*) if $\hat{H}^* > H^*$ (resp., $\hat{H}^* < H^*$); it is *CS-neutral* if $\hat{H}^* = H^*$. Similarly, we say that a merger is *W-increasing* if it raises social welfare, etc. Let T^M denote the merged firm's post-merger type, which takes into account any possible merger-specific (in-)efficiencies in scope, marginal costs, and qualities.

Consumer surplus and social welfare effects. The following proposition provides necessary and sufficient conditions for a merger to be CS-increasing (resp. W-increasing):

Proposition XIV. *There exist two cutoffs* $\hat{T}^M, \tilde{T}^M > 0$ *such that merger* M *is*

- *CS-neutral if* $T^M = \hat{T}^M$*, CS-increasing if* $T^M > \hat{T}^M$*, and CS-decreasing if* $T^M < \hat{T}^M$*;*
- *W-neutral if* $T^M = \tilde{T}^M$*, W-increasing if* $T^M > \tilde{T}^M$*, and W-decreasing if* $T^M < \tilde{T}^M$*.*

Moreover, $\hat{T}^M > \max\left(\tilde{T}^M, \sum_{f\in\mathcal{I}} T^f\right)$*. If* M *is CS-nondecreasing, it is profitable in that it raises the joint profit of the merger partners.*

Proof. See Section XV.3.1. □

Inequality $\hat{T}^M > \sum_{f\in\mathcal{I}} T^f$ means that, for a merger to be CS-nondecreasing, the merger has to involve synergies, as in Williamson (1968) and Farrell and Shapiro (1990). To see why a CS-nondecreasing merger is profitable, consider first a CS-neutral merger. As such a merger involves synergies and does not change the equilibrium value of the aggegator, it must be profitable.[27] By Proposition 6, merger M is even more profitable if it is CS-increasing rather than CS-neutral since it involves larger synergies.[28] In contrast to Farrell and Shapiro (1990), we are also able to exploit the monotonicity of social welfare in firms'

[27] Note that a merger that does not involve any synergies is profitable as well, as in Deneckere and Davidson (1985). To see this, note that such a merger implies a downward shift in the aggregate fitting-in function (since the merging parties internalize competitive externalities), and therefore lowers the equilibrium aggregator level. Since $m'(\cdot) > 0$, the outsiders respond to the merger by setting higher prices. This positive indirect effect reinforces the direct effect of the merger.

[28] The insight that CS-nondecreasing mergers are profitable was first obtained by Nocke and Whinston (2010) in the context of the homogeneous-goods Cournot model.

types to establish the existence of a cutoff type $\tilde{T}^M$ such that the merger is W-increasing if and only if $T^M > \tilde{T}^M$. Inequality $\hat{T}^M > \tilde{T}^M$ follows immediately from the fact that a CS-neutral merger is profitable.[29]

External effects. The aggregative structure of our pricing game allows us to extend Farrell and Shapiro (1990)'s analysis of the external effects of a merger. To the extent that a merger is proposed by the merger partners only if it is in their joint interest to do so, a positive external effect is a sufficient ("safe harbor") condition for the merger to raise social welfare. The idea behind focusing on the external effect is that the profitability of a merger depends on the magnitude of internal cost savings, and that these are hard to assess for an antitrust authority.

The external effect of the merger, defined as the sum of its impact on consumer surplus and outsiders' profits, is given by (recall that $\pi(\cdot) = m(\cdot) - 1$):

$$\log \hat{H}^* - \log H^* + \sum_{f \in \mathcal{O}} \left(m\left(\frac{T^f}{\hat{H}^*}\right) - m\left(\frac{T^f}{H^*}\right)\right) = -\int_{H^*}^{\hat{H}^*} \frac{\eta(H)}{H} dH,$$

where

$$\eta(H) \equiv -1 + \sum_{f \in \mathcal{O}} \frac{T^f}{H} m'\left(\frac{T^f}{H}\right).$$

Hence, as in Farrell and Shapiro (1990), the merger can be thought of as a sequence of infinitesimal mergers dH, where, along the sequence, the value of the aggregator changes progressively from H^* to $\hat{H}^*$. The sign of the external effect of an infinitesimal CS-decreasing (resp. CS-increasing) merger is thus given by $\eta(H)$ (resp. $-\eta(H)$). In the following, we focus on CS-decreasing mergers to fix ideas.

An infinitesimal CS-decreasing merger $dH < 0$ reduces consumer surplus by dH/H, which corresponds to the first term in the definition of η. It also increases the profit of every outsider $f \in \mathcal{O}$ by dH/H times $(T^f/H)m'(T^f/H)$. Defining $\phi_\alpha(S(T^f/H)) \equiv (T^f/H)m'(T^f/H)$ with $\alpha = (\sigma - 1)/\sigma$ in the CES case and $\alpha = 1$ in the MNL case, we show in Section XV.3.2 that this profit change can be rewritten as:

$$\phi_\alpha(s) = \frac{\alpha s(1-s)}{(1-\alpha s)(1-s+\alpha s^2)}, \quad \forall s \in (0,1).$$

Hence, the external effect of an infinitesimal CS-decreasing merger is given by

$$\eta(H)\frac{dH}{H} = \left(-1 + \sum_{f \in \mathcal{O}} \phi_\alpha(s^f)\right)\frac{dH}{H}.$$

[29] Whether or not $\tilde{T}^M > \sum_{f \in \mathcal{I}} T^f$ is unclear. On the one hand, a merger that does not involve synergies lowers the equilibrium aggregator level. On the other hand, it reallocates market shares toward the outsiders, which can raise social welfare if the outsiders are initially producing too little.

Computing the external effect of an infinitesimal merger therefore only requires knowledge of the outsiders' market shares and of the value of the demand-side parameter α.

To go further, we study the behavior of the function $\phi_\alpha(\cdot)$:

Proposition XV. *There exists $\bar{\alpha}(\simeq 0.82)$ such that:*

(i) If $\alpha \leq \bar{\alpha}$, then the external effect of any CS-decreasing merger is negative.

(ii) If $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have positive external effects, and CS-increasing mergers that have negative external effects.

Moreover, if $\alpha > \bar{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0, 1]$ and $\hat{s}(\alpha) \in (0, 1)$ such that:

(iii) $s \mapsto \phi_\alpha(s)$ is strictly increasing on $(0, s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha), 1)$.[30]

(iv) $s \mapsto \phi_\alpha(s)$ is strictly convex on $(0, \hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha), 1)$.[31]

Proof. See Section XV.3.2. □

Part (i) of the proposition implies that, if $\alpha < \bar{\alpha}$ (i.e., demand is of the CES form and $\sigma < \bar{\sigma} \simeq 5.53$), then the positive effect of the CS-decreasing merger on outsiders' profits is always outweighed by the negative effect on consumer surplus, implying a negative external effect.

Next, suppose that $\alpha > \bar{\alpha}$ (i.e., demand is either of the MNL form or of the CES form with $\sigma > \bar{\sigma}$). Then, by part (ii) of the proposition, there exist CS-decreasing mergers that have positive (resp. negative) external effects. Parts (iii) and (iv) provide conditions under which the external effect of an infinitesimal CS-decreasing merger is more likely to be positive. To understand these conditions, note that the decrease in H has two effects on an outsider's profit. First, holding fixed outsiders' markups, it increases the profit of each outsider f by $\Pi^f \times |dH/H|$. Hence, the "direct" effect on outsiders' joint profit is proportional to their joint profit. Second, outsiders respond by increasing their markups.

To grasp the intuition for parts (iii) and (iv), it is useful to focus on the first, direct effect. By part (iii), the merger is more likely to have a positive external effect if the outsiders have high market shares (provided no outsider has a market share above $s^*(\alpha)$). The intuition is that if outsiders command larger market shares, they make larger profits, and therefore benefit more from the direct effect of the reduction in H. By part (iv), the merger is more likely to have a positive external effect if the outsiders' market shares are more concentrated (provided no outsider has a market share above $\hat{s}(\alpha)$).[32] The intuition is that if outsiders'

[30] In the MNL case, $s^*(\alpha) = 1$; in the CES case, numerically, we find that $s^*(\alpha) \geq 0.68$ for every α.

[31] Numerically, we find that $\hat{s}(\alpha) \geq 0.28$ for every α.

[32] We say that the profile of market shares s' is *more concentrated* than the profile s if the cumulative distribution function of s' second-order stochastically dominates that of s. See Section XV.3.2.

market shares are more concentrated, their joint profit is larger, implying that they jointly benefit more from the direct effect of the reduction in H.[33]

Part (iv) implies that relying on the pre-merger Herfindahl-Hirschman index (HHI) to evaluate the social desirability of a merger can be misguided. To see this, consider two industries, and suppose that the vector of insiders' market shares is the same in both industries. Suppose also that outsiders' market shares are more concentrated in the first industry than in the second. Then, the first industry's HHI is higher than the second's. However, the merger in the first industry is more likely to have a positive external effect than the one in the second industry.

The external effect of the non-infinitesimal CS-decreasing merger M is the integral of the external effects of the infinitesimal mergers along the path from H^* to $\hat{H}^* < H^*$. If products are relatively poor substitutes ($\alpha < \bar{\alpha}$), then, by part (i) of Proposition XV, merger M has a negative external effect. Suppose instead that products are relatively good substitutes ($\alpha > \bar{\alpha}$). As the merger is CS-decreasing by assumption, outsiders' market shares increase along the sequence of infinitesimal mergers from H^* to $\hat{H}^*$. Hence, if $\eta(H^*) > 0$ (i.e., at the pre-merger aggregator level, an infinitesimal CS-decreasing merger has a positive external effect), then, by part (iii) of Proposition XV, $\eta(H)$ remains positive along the sequence (provided no outsider reaches a market share larger than s^*), and so the external effect of the merger is positive. But to check whether $\eta(H^*) > 0$ involves using only the outsiders' *pre*-merger market shares.

XV.2 Dynamic Merger Analysis

In many industries, mergers are not isolated events. Evaluating a proposed merger on the basis of current market conditions therefore appears inappropriate. However, future market conditions are affected by today's decision on the proposed merger, for two reasons. First, today's decision changes the welfare effects of potential future mergers and therefore the set of mergers that will be approved in the future. Second, today's decision changes the profitability of potential future mergers and therefore the set of mergers that will be proposed in the future.

For the case of Cournot competition with homogeneous goods, Nocke and Whinston (2010) show that this problem has a surprisingly simple solution when the authority's objective is to maximize discounted consumer surplus: A myopic merger approval policy that approves a proposed merger today if and only if it does not lower current consumer surplus (completely ignoring the possibility of future mergers) is dynamically optimal.[34] They

[33]The reason why this intuition may not hold if some of the outsiders are too large is the result of the second, indirect effect. Holding H fixed, the induced increase in an outsider's markup decreases its profit. This holds since oligopolistic markups are always above those of monopolistically competitive firms that perceive H as fixed, so any further increase must reduce profit for a fixed H. The qualifiers in parts (iii) and (iv) arise because the effect of a given increase in the markup is heterogeneous across outsiders, as is the extent of the induced markup increase.

[34]Dynamic optimality obtains in a strong sense: The antitrust authority could not improve upon this outcome even if it had perfect foresight about future merger possibilities (which, by assumption, it does not)

consider a T-period model in which merger opportunities arise stochastically over time and, in each period, firms with a feasible but not-yet-approved merger have to decide whether to propose it and the antitrust authority has to decide which of the proposed mergers to approve (if any). In addition to Cournot competition, the two key assumptions are: First, the set of potential mergers is disjoint (i.e., no firm can be party to more than one potential merger). Second, rejected mergers can be proposed again in the future (i.e., merger opportunities do not disappear).

As we show in Section XV.3.4, the optimality result carries over to the case of price competition between multiproduct firms when demand takes either the CES or MNL forms. The optimality result comes in two parts. First, assuming that all feasible and not-yet-approved mergers are proposed in every period, a myopic merger approval policy is dynamically optimal in that it maximizes the discounted sum of consumer surplus. Second, if the antitrust authority adopts a myopic merger approval policy, then in any subgame-perfect equilibrium, every merger that the authority would want to approve in the dynamically optimal solution will be proposed.

The key observation for the first part of the optimality result is the following:

Proposition XVI. *For any merger M, the post-merger cutoff type $\hat{T}^M$ is decreasing in the pre-merger value H^* of the aggregator.*

Proof. See Section XV.3.3. □

The intuition is straightforward: An increase in the aggregator reduces the market share of the merger partners and thereby the market power effect of the merger. The proposition implies a certain sign-preserving complementarity in the consumer surplus effects of mergers. Suppose mergers M_1 and M_2 are both CS-nondecreasing in isolation (i.e., given the pre-merger aggregator H^*). Then, each merger M_i remains CS-nondecreasing after the other merger M_{-i} has been implemented. Conversely, suppose both mergers are CS-decreasing in isolation. Then, each remains CS-decreasing after the other one has been implemented. The proposition also implies that a CS-decreasing merger M_2 may become CS-nondecreasing once a CS-increasing merger M_1 has been implemented; if so, M_1 remains CS-increasing conditional on M_2 taking place.[35]

The complementarity result implies that if the antitrust authority approves only mergers that are CS-nondecreasing at the time of approval, then it will never have ex-post regret about previously approved mergers as these will always remain CS-nondecreasing given the set of approved mergers. In conjunction with the assumption that merger opportunities do not disappear, the complementarity also implies that the authority will never have ex-post regret about having blocked mergers that were CS-decreasing at the time of the decision since these mergers can (and will) be proposed and approved once they become CS-nondecreasing.

nor if it could undo previously approved mergers (which, by assumption, it cannot).

[35]To see this, note that if M_1 is implemented before M_2, then consumer surplus increases at each step. Hence, conditional on implementing M_2 (which, by assumption, decreases consumer surplus), M_1 must increase consumer surplus.

Turning to the second part of the optimality result, Proposition XIV implies that any CS-nondecreasing merger M is profitable at the time of its approval. Moreover, if the authority adopts a CS-based merger approval policy, the merger remains CS-nondecreasing and therefore profitable given the set of other mergers that will be approved along the equilibrium path. Surprisingly, a CS-nondecreasing merger M remains profitable even if it induces (directly or indirectly) the implementation of additional mergers, all of which are CS-nondecreasing at the time of their approval (and therefore reduce the profit of the merged entity M).[36] As a result, a CS-based myopic merger approval policy solves the moral hazard problem arising from the fact that the authority can approve only mergers that are proposed in the first place: In any equilibrium, any merger that the authority would like to approve will be proposed.

XV.3 Proofs

XV.3.1 Static Merger Analysis: Proof of Proposition XIV

Proof. Let

$$\hat{T}^M \equiv H^* S^{-1}\left(\sum_{f\in\mathcal{I}} S\left(\frac{T^f}{H^*}\right)\right). \tag{li}$$

$\hat{T}^M$ is well-defined, since S is strictly increasing and has range $(0,1)$.

If $T^M = \hat{T}^M$, we have:

$$1 = \frac{H^0}{H^*} + \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*}\right) = \frac{H^0}{H^*} + S\left(\frac{T^M}{H^*}\right) + \sum_{f\in\mathcal{O}} S\left(\frac{T^f}{H^*}\right),$$

where the first equality is the pre-merger equilibrium condition whereas the second equality follows from $T^M = \hat{T}^M$. Therefore, $\hat{H}^* = H^*$, i.e., the merger is CS-neutral if $T^M = \hat{T}^M$. As $S'(\cdot) > 0$, if $T^M > \hat{T}^M$, we have

$$\frac{H^0}{H^*} + S\left(\frac{T^M}{H^*}\right) + \sum_{f\in\mathcal{O}} S\left(\frac{T^f}{H^*}\right) > 1,$$

implying that $\hat{H}^* > H^*$, so the merger is CS-increasing. Similarly, if $T^M < \hat{T}^M$, then $\hat{H}^* < H^*$, so the merger is CS-decreasing.

Next, we note that a CS-neutral merger involves synergies in that $\hat{T}^M > \sum_{f\in\mathcal{I}} T^f$. Suppose otherwise that $\hat{T}^M \leq \sum_{f\in\mathcal{I}} T^f$. Then,

$$S\left(\frac{\hat{T}^M}{H^*}\right) \leq S\left(\sum_{f\in\mathcal{I}} \frac{T^f}{H^*}\right) < \sum_{f\in\mathcal{I}} S\left(\frac{T^f}{H^*}\right),$$

where the first inequality follows from $S'(\cdot) > 0$ and the second from Lemma XXVI. But

[36] A merger proposal can only increase, not decrease the set of other mergers that will be implemented.

then the merger would be CS-decreasing, a contradiction. Hence, $\hat{T}^M > \sum_{f\in\mathcal{I}} T^f$.

We now show that a CS-neutral merger is profitable. Recall that, under CES demands, $\pi = m - 1$ and $S = \frac{\sigma}{\sigma-1}\frac{m-1}{m}$. It follows that $\pi = \frac{\sigma-1}{\sigma} mS$. Similarly, under MNL demands, $\pi = mS$. In both cases, π is proportional to mS. Note that

$$m\left(\frac{T^M}{H^*}\right) S\left(\frac{T^M}{H^*}\right) = m\left(\frac{T^M}{H^*}\right)\sum_{f\in\mathcal{I}} S\left(\frac{T^f}{H^*}\right) > \sum_{f\in\mathcal{I}} m\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right),$$

where the equality follows because the merger is CS-neutral, and the inequality follows because $\hat{T}^M > T^f$ for every $f \in \mathcal{I}$ and $m'(\cdot) > 0$. Hence, merger M is profitable if $T^M = \hat{T}^M$. Next, suppose that the merger is CS-increasing, i.e., $T^M > \hat{T}^M$. Then, by Proposition 6, the merged firm makes a strictly higher equilibrium profit than when its type is $\hat{T}^M$. This implies in particular that the merger is profitable.

Finally, we establish the existence of threshold $\tilde{T}^M$. Note first that, if $T^M = \hat{T}^M$, then the merger is W-increasing, since it raises the joint profits of the merging parties, but affects neither consumer surplus, nor the outsiders' profits. On the other hand, it is straightforward to show that, as T^M tends to zero, $\bar{H}^*$ converges to the equilibrium aggregator level which would prevail if only the outsiders were present. Social welfare in that case is equal to the limit of social welfare pre-merger as T^f tends to 0 for every $f \in \mathcal{I}$, which, by monotonicity, is strictly lower than equilibrium social welfare when $T^f > 0$ for every $f \in \mathcal{I}$. Therefore, the merger is W-decreasing if T^M is low enough. By the intermediate value theorem, there exists $\tilde{T}^M$ such that the welfare is W-neutral if $T^M = \tilde{T}^M$. By monotonicity of social welfare, the merger is W-increasing if $T^M > \tilde{T}^M$, and W-decreasing if $T^M < \tilde{T}^M$. □

XV.3.2 Static Merger Analysis: External Effects

We first derive formulas for η:

Lemma XXVIII. *$\eta(H)$ is given by:*

$$\eta(H) = -1 + \sum_{f\in\mathcal{O}} \phi_\alpha(s^f),$$

where $\alpha = (\sigma-1)/\sigma$ in the CES case, $\alpha = 1$ in the MNL case, $s^f = S(T^f/H)$, and

$$\phi_\alpha(s) = \frac{\alpha s(1-s)}{(1-\alpha s)(1-s+\alpha s^2)}, \quad \forall s \in (0,1).$$

Proof. This follows from the definition of η and from the fact that

$$\begin{aligned} xm'(x) &= x\alpha\frac{S'(x)}{(1-\alpha S(x))^2}, \text{ since } m(x) = \frac{1}{1-\alpha S(x)}, \\ &= \frac{\alpha}{(1-\alpha S(x))^2}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}, \text{ using equation (xlvi),} \end{aligned}$$

$$= \frac{\alpha S(x)(1 - S(x))}{(1 - \alpha S(x))(1 - S(x) + \alpha S(x)^2)},$$
$$= \phi_\alpha(S(x)).$$
□

Next, we study the properties of the function $\phi_\alpha(\cdot)$:

Lemma XXIX. *Function $(s, \alpha) \mapsto \phi_\alpha(s)$ has the following properties:*

(a) For every $s \in (0, 1)$, $\alpha \mapsto \phi_\alpha(s)$ is strictly increasing.

There exists $\hat{\alpha} \in (0, 1)$ such that:

(b) If $\alpha \leq \hat{\alpha}$, then $\phi_\alpha(s) \leq s$ for every $s \in [0, 1]$.

(c) If $\alpha > \hat{\alpha}$, then there exist $0 \leq \underline{s}(\alpha) < \overline{s}(\alpha) \leq 1$ such that, for every $s \in [0, 1]$, $\phi_\alpha(s) > s$ if and only if $s \in (\underline{s}(\alpha), \overline{s}(\alpha))$.

Moreover, if $\alpha > \hat{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0, 1]$ and $\hat{s}(\alpha) \in (0, 1)$ such that:[37]

(d) $s \mapsto \phi_\alpha(s)$ is strictly increasing on $(0, s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha), 1)$.

(e) $s \mapsto \phi_\alpha(s)$ is strictly convex on $(0, \hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha), 1)$.

Proof. We prove the lemma (analytically) using Mathematica. Mathematica files are available upon request. □

The following lemma is the final step toward proving proposition XV:

Lemma XXX. *Let $\bar{\alpha} = \frac{3}{2}(\sqrt{57} - 7) \simeq 0.82$. If $\alpha \leq \bar{\alpha}$, then any infinitesimal CS-decreasing merger has a negative external effect. If instead $\alpha > \bar{\alpha}$, then there exist infinitesimal CS-decreasing mergers that have positive external effects, and infinitesimal CS-increasing mergers that have negative external effects.*

Proof. Define

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}^n, \text{ where } \mathcal{S}^n = \{s \in [0, 1]^n : \sum_{i=1}^n s_i \leq 1\} \ \forall n \geq 1,$$
$$\bar{\mathcal{S}} = \bigcup_{n \geq 1} \mathcal{S}^n, \text{ where } \bar{\mathcal{S}}^n = \{s \in [0, 1]^n : \sum_{i=1}^n s_i = 1\} \ \forall n \geq 1,$$

[37] More on thresholds $\underline{s}(\alpha)$, $\overline{s}(\alpha)$, $s^*(\alpha)$ and $\hat{s}(\alpha)$:

- In the MNL case, $\underline{s}(1) = 0$ and $\overline{s}(1) = 1$. Otherwise, both thresholds are interior.
- In the MNL case, $s^*(1) = 1$. Otherwise, $0.68 \leq s^*(\alpha) < 1$.
- $0.28 \leq \hat{s}(\alpha) < 1$.

and[38]

$$\Psi(\alpha) = \sup_{s\in\mathcal{S}} \sum_i \phi_\alpha(s_i),\ \forall \alpha \in (\hat{\alpha}, 1].$$

Clearly, since $\phi_\alpha(s) \geq 0$ for all s, we have that $\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}} \sum_i \phi_\alpha(s_i)$. Next, we claim that $\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}^4} \sum_{i=1}^4 \phi_\alpha(s_i)$. To prove this, we show that, for every $s \in \bar{\mathcal{S}}$, there exists $s' \in \bar{\mathcal{S}}^4$ such that

$$\sum_i \phi_\alpha(s_i) \leq \sum_{i=1}^4 \phi_\alpha(s'_i).$$

If s belongs to $\mathcal{S}^n$ for some $n \leq 4$, or, more generally, if s has at most four components different from zero, then this is obvious. Assume instead that s has five or more components different from zero. Assume without loss of generality that $s \in \bar{\mathcal{S}^n}$ for some $n \geq 5$, that $s_i > 0$ for every i, and that the components of s_i have been sorted in increasing order. We construct s' by induction.

Let us first define a function ξ, which takes as argument a profile of market shares $s \in \bar{\mathcal{S}^n}$ sorted in increasing order and with strictly positive components, and returns a profile of market shares $\xi(s)$ sorted in increasing order and with strictly positive components, such that either $\xi(s) \in \bar{\mathcal{S}^n}$, or $\xi(s) \in \overline{\mathcal{S}^{n-1}}$. ξ is defined as follows:

- If $s_2 \geq \hat{s}(\alpha)$ (or if $s \in \mathcal{S}^1$), then $\xi(s) = s$.
- If $s_2 < \hat{s}(\alpha)$, then do the following:
  - If $s_1 + s_2 \leq \hat{s}(\alpha)$, then form the $(n-1)$-dimensional vector with first component $s_1 + s_2$ and remaining components $(s_i)_{3\leq i\leq n}$, and sort that vector in increasing order to obtain $\xi(s)$.
  - If instead $s_1 + s_2 > \hat{s}(\alpha)$, then form the n-dimensional with first component $s_1 + s_2 - \hat{s}(\alpha)$, second component $\hat{s}(\alpha)$, and remaining components $(s_i)_{3\leq i\leq n}$, and sort that vector in increasing order to obtain $\xi(s)$.

We now argue that, for every $\tilde{s} \in \bar{\mathcal{S}}$,

$$\sum_i \phi_\alpha(\tilde{s}_i) \leq \sum_i \phi_\alpha\left(\xi_i(\tilde{s})\right).$$

If $\xi(\tilde{s}) = \tilde{s}$, then this is obvious. Suppose $\xi(\tilde{s}) \neq \tilde{s}$, i.e., $s_2 < \hat{s}(\alpha)$. If $s_1 + s_2 \leq \hat{s}(\alpha)$, then

$$\sum_i \phi_\alpha\left(\xi_i(\tilde{s})\right) - \sum_i \phi_\alpha(\tilde{s}_i) = \phi_\alpha(s_1 + s_2) - \phi_\alpha(s_1) - \phi_\alpha(s_2),$$

[38]Notation: Let $s \in \mathcal{S}$ and $n \geq 1$ such that $s \in \mathcal{S}^n$. We write

$$\sum_i \phi_\alpha(s_i) = \sum_{i=1}^n \phi_\alpha(s_i).$$

$$= \int_0^{s_1} (\phi'_\alpha(s_2+t) - \phi'_\alpha(t))\, dt,$$
$$\geq 0,$$

since $\phi_\alpha(\cdot)$ is convex on $[0, \hat{s}(\alpha)]$. If instead $s_1 + s_2 > \hat{s}(\alpha)$, then

$$\sum_i \phi_\alpha(\xi_i(\tilde{s})) - \sum_i \phi_\alpha(\tilde{s}_i) = \phi_\alpha(\hat{s}(\alpha)) + \phi_\alpha(s_1 + s_2 - \hat{s}(\alpha)) - \phi_\alpha(s_1) - \phi_\alpha(s_2),$$
$$= \int_0^{\hat{s}(\alpha)-s_2} (\phi'_\alpha(s_2+t) - \phi'_\alpha(s_1 - t))\, dt,$$
$$\geq 0,$$

where the inequality follows again by convexity of $\phi_\alpha(\cdot)$ on $[0, \hat{s}(\alpha)]$.

We can now define the sequence $(s^k)_{k\geq 0}$ by induction: $s^0 = s$; $s^{k+1} = \xi(s^k)$ for every $k \geq 0$. Let m^k denote the number of components of s^k greater or equal to $\hat{s}(\alpha)$, and n^k denote the dimensionality of the vector s^k. By definition of ξ and of the sequence $(s^k)_{k\geq 0}$, the sequence of integers $(m^k)_{k\geq 0}$ (resp. $(n^k)_{k\geq 0}$) is non-decreasing (resp. non-increasing) and bounded above by n (resp. bounded below by 1). Therefore, those sequences of integers are eventually stationary: There exists $K \geq 0$ such that $m^k = m^{k+1}$ and $n^k = n^{k+1}$ for every $k \geq K$. It follows that $(s^k)_{k\geq 0}$ is also stationary after K. Let s' be the stationary value of the sequence $(s^k)_{k\geq 0}$. Then, by induction on k,

$$\sum_i \phi_\alpha(s_i) \leq \sum_i \phi_\alpha(s'_i).$$

Moreover, s' has at most one component in $[0, \hat{s}(\alpha))$ (for otherwise, $\xi(s')$ would not be equal to s'). Let n' be the dimensionality of the vector s'. Then,

$$1 = \sum_{i=1}^{n'} s'_i \geq (n'-1)\hat{s}(\alpha) \geq 0.28 \times (n'-1),$$

where the last inequality follows by Lemma XXIX (see footnote 37). It follows that $n' \leq 4$. Having constructed s', we can conclude that

$$\Psi(\alpha) = \sup_{s \in \bar{\mathcal{S}}^4} \sum_{i=1}^{n} \phi_\alpha(s_i). \tag{lii}$$

By continuity of ϕ_α and by compactness of $\bar{\mathcal{S}}^4$, the maximization problem defined by (lii) has a solution. Let s be such a solution. Then, by the convexity argument used in the construction of s', s has a most one component in $(0, \hat{s}(\alpha))$. Moreover, since ϕ_α is strictly concave on $[\hat{s}(\alpha), 1]$, the components of s that are greater or equal to $\hat{s}(\alpha)$ must be equal to each other. It follows that

$$\Psi(\alpha) = \max_{x\in[0,1]} \max\left(\phi_\alpha(x)+\phi_\alpha(1-x), \phi_\alpha(x)+2\phi_\alpha\left(\frac{1-x}{2}\right), \phi_\alpha(x)+3\phi_\alpha\left(\frac{1-x}{3}\right)\right).$$

We (analytically) solve the above maximization problem using Mathematica. We obtain:

$$\Psi(\alpha) = \begin{cases} \frac{18\alpha}{18-3\alpha-\alpha^2} & \text{if } \alpha \le \frac{6}{7}, \\ \frac{4\alpha}{4-\alpha^2} & \text{otherwise.} \end{cases}$$

It is straightforward to check that Ψ is strictly increasing, and that $\Psi(\hat{\alpha}) < 1 < \Psi(1)$. The unique solution of equation $\Psi(\alpha) = 1$ on interval $(\hat{\alpha}, 1]$ is $\bar{\alpha} = \frac{3}{2}(\sqrt{57}-7)$.

We can conclude. Assume first that $\alpha \le \hat{\alpha}$. Then, by Lemma XXIX-(b), $\sum_{f\in\mathcal{O}} \phi_\alpha(s^f) \le \sum_{f\in\mathcal{O}} s^f < 1$ for every profile of outsiders' market shares $(s^f)_{f\in\mathcal{O}}$. Therefore, any infinitesimal CS-decreasing merger must have a negative external effect.

Next, assume that $\alpha \in (\hat{\alpha}, \bar{\alpha}]$. Then, for every profile of outsiders' market shares $(s^f)_{f\in\mathcal{O}}$,

$$\sum_{f\in\mathcal{O}} \phi_\alpha(s^f) < \phi_\alpha(1-s^f) + \sum_{f\in\mathcal{O}} \phi_\alpha(s^f) \le \Psi(\alpha) \le \Psi(\bar{\alpha}) = 1.$$

Therefore, any infinitesimal CS-decreasing merger must have a negative external effect.

Finally, assume $\alpha > \bar{\alpha}$. We first show that there exists an infinitesimal CS-decreasing merger that has a negative external effect. Let $\mathcal{O} = \{1\}$ and $\mathcal{I} = \{2,3\}$. Since $\phi_\alpha(\cdot)$ is continuous and $\phi_\alpha(0) = 0$, there exists $s \in (0,1)$ such that $\phi_\alpha(s) < 1$. Let $T^1 = S^{-1}(s)$, and $T^2 = T^3 = S^{-1}\left(\frac{1-s}{2}\right)$. Then, by construction, the pre-merger equilibrium aggregator level is $H = 1$, and market shares are as follows: $s^1 = s$, $s^2 = s^3 = \frac{1-s}{2}$. The external effect of an infinitesimal and CS-decreasing merger between firms 2 and 3 is given by $\phi_\alpha(s) - 1$, which is strictly negative by construction. Next, we claim that there exists an infinitesimal CS-decreasing merger that has a positive external effect. Since $\Psi(\alpha) > 1$, there exists $(s_i)_{1\le i\le n} \in (0,1]^n$ such that $\sum_{i=1}^n s_i \le 1$ and $\sum_{i=1}^n \phi_\alpha(s_i) > 1$. By continuity, for $\varepsilon > 0$ small enough, $\sum_{i=1}^n \phi_\alpha(s_i - \varepsilon) > 1$. Let $\mathcal{O} = \{1, \ldots, n\}$, $\mathcal{I} = \{n+1, n+2\}$, $s'^i = s_i - \varepsilon$ for every $i \in \mathcal{O}$, $s'^i = \frac{1}{2}\left(1 - \sum_{j=1}^n s'^j\right)$ for $i \in \mathcal{I}$, and $T^i = S^{-1}(s'^i)$ for every $i \in \mathcal{I} \cup \mathcal{O}$. Then, by construction, an infinitesimal and CS-decreasing merger between the insiders has a positive external effect.

Since any CS-decreasing merger can be decomposed into the integral of infinitesimal CS-decreasing mergers, and since a CS-decreasing merger can be made infinitesimal by tweaking the post-merger type of the merged entity, the above results extend immediately to non-infinitesimal mergers: If $\alpha \le \bar{\alpha}$, then any CS-decreasing merger has a negative external effect; If $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have positive external effects, and CS-decreasing mergers that have negative external effects. $\square$

Proposition XV follows immediately from Lemmas XXIX and XXX.

Finally, we formalize and prove our statements on the impact of the concentration of outsiders' market shares. Fix $\alpha > \bar{\alpha}$. Assume without loss of generality that $\mathcal{O} = \{1, \ldots, n\}$ with $n \geq 2$. An outsider industry structure is a vector of outsiders' market shares $\mathbf{s} \in [0, \hat{s}(\alpha)]^n$ such that $\sum_{i=1}^n s_i < 1$. To every outsider industry structure $\mathbf{s}$, we associate a discrete probability distribution $P_{\mathbf{s}}(\cdot)$, which is defined as follows:

$$P_{\mathbf{s}}(x) = \frac{1}{n} \left| \{ i \in \{1, \ldots, n\} : \quad s_i = x \} \right|, \quad \forall x \in \mathbb{R}.$$

Note that the mean of the probability distribution $P_{\mathbf{s}}$ is equal to $\frac{\sum_{i=1}^n s_i}{n}$.

We now use these associated probability distributions to define a partial order on the set of outsider industry structures. We say that the outsider industry structure $\mathbf{s}'$ is more concentrated than the outsider industry structure $\mathbf{s}$ if $P_{\mathbf{s}}$ and $P_{\mathbf{s}'}$ have the same mean (i.e., the aggregate market shares of the outsiders are the same in both industry structures) and $P_{\mathbf{s}}$ second-order stochastically dominates $P_{\mathbf{s}'}$. For instance, with $n = 2$, the industry structure $(0.05, 0.15)$ is more concentrated than the industry structure $(0.1, 0.1)$.

Suppose that the outsider industry structure $\mathbf{s}'$ is more concentrated than the outsider industry structure $\mathbf{s}$. Then, since ϕ_α is convex on a set which contains the supports of $P_{\mathbf{s}}(x)$ and $P_{\mathbf{s}'}(x)$,

$$\int_{\mathbb{R}} \phi_\alpha(x) dP_{\mathbf{s}'}(x) \geq \int_{\mathbb{R}} \phi_\alpha(x) dP_{\mathbf{s}}(x).$$

Using the definition of $P_{\mathbf{s}}$ and $P_{\mathbf{s}'}$, we obtain:

$$\sum_{i=1}^n \frac{1}{n} \phi_\alpha(s_i') \geq \sum_{i=1}^n \frac{1}{n} \phi_\alpha(s_i).$$

Therefore, η is higher with outsider industry structure $\mathbf{s}'$ than with outsider industry structure $\mathbf{s}$. This implies that the external effect of an infinitesimal CS-decreasing merger is more likely to be positive when the outsiders have more concentrated market shares. Note that, by convexity of the function $x \mapsto x^2$, the industry HHI is higher under industry structure $\mathbf{s}'$ than under industry structure $\mathbf{s}$.

XV.3.3 Dynamic Merger Analysis: Proof of Proposition XVI

Proof. Let $\mathcal{I}$ be the set of insiders associated with merger M. Differentiating equation (li), we obtain

$$\begin{aligned} S'\left(\frac{\hat{T}^M}{H^*}\right) \frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*} S'\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f \in \mathcal{I}} \frac{T^f}{H^*} S'\left(\frac{T^f}{H^*}\right), \\ &= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right) S\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f \in \mathcal{I}} \varepsilon\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right), \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)\sum_{f\in\mathcal{I}} S\left(\frac{T^f}{H^*}\right) - \sum_{f\in\mathcal{I}} \varepsilon\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right), \\
&= \sum_{f\in\mathcal{I}} \left(\varepsilon\left(\frac{\hat{T}^M}{H^*}\right) - \varepsilon\left(\frac{T^f}{H^*}\right)\right) S\left(\frac{T^f}{H^*}\right), \\
&< 0,
\end{aligned}
$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma XXV and from the fact that $\hat{T}^M > T^f$ for every $f \in \mathcal{I}$. □

XV.3.4 Dynamic Merger Analysis: Dynamic Optimality of Myopic Merger Approval Policy

Consider two mergers, M_1 and M_2, and assume that these mergers are disjoint, i.e., no firm takes part in both.

Proposition XVII. *If merger M_i is CS-nondecreasing (and hence profitable) in isolation, it remains CS-nondecreasing (and hence profitable) if another merger M_j, $j \neq i$, that is CS-nondecreasing in isolation takes place. If merger M_i is CS-decreasing in isolation, it remains CS-decreasing if another merger M_j, $j \neq i$, that is CS-decreasing in isolation takes place.*

Proof. Suppose M_i is CS-nondecreasing in isolation, which means that $T^{M_1} \geq \hat{T}^{M_1}$. If the CS-nondecreasing merger M_j takes place, the equilibrium value of the aggregator H^* weakly increases, and so – by Proposition XVI – the cutoff $\hat{T}^{M_1}$ weakly decreases. As T^{M_1} was initially above the cutoff, it therefore remains so after M_j has taken place, i.e., M_i is still CS-nondecreasing. A similar argument can be used to show the sign-preserving complementarity for mergers that are CS-decreasing in isolation. The assertion on profitability follows from Proposition XIV. □

Proposition XVIII. *Suppose that merger M_1 is CS-nondecreasing in isolation whereas merger M_2 is CS-decreasing in isolation but CS-nondecreasing once merger M_1 has taken place. Then, merger M_1 is CS-increasing (and hence profitable), conditional on merger M_2 taking place. Moreover, the joint profit of the firms involved in M_1 is strictly larger if both mergers take place than if neither does.*

Proof. As in the proof of Proposition 2 in Nocke and Whinston (2010), reverse the order of the two mergers: Consider first implementing merger M_2 (step 1) and then merger M_1 (step 2). As consumer surplus must, by assumption, be (weakly) higher after both mergers have taken place than before, and because consumer surplus (strictly) falls at step 1 (again, by assumption), consumer surplus must (strictly) increase at step 2. That is, M_1 is CS-increasing, conditional on M_2 taking place. By Proposition XIV, this implies that the joint profit of the firms in M_1 must go up at step 2. The joint profit of the firms in M_1 must go up at step 1 as well, as the CS-decreasing merger at step 1 induces a reduction in the equilibrium value of the aggregator, which benefits all outsiders to that merger by Proposition 6. □

We now embed our pricing game in a dynamic model with endogenous mergers and merger policy, as in Nocke and Whinston (2010). There are T periods, and a set $\{M_1, M_2, ..., M_K\}$ of disjoint potential mergers. Merger M_k becomes feasible at the beginning of period t with probability $p_{kt} \in [0, 1]$, where $\sum_t p_{kt} \leq 1$. Conditional on becoming feasible, the post-merger type of the merged firm M_k is drawn from some distribution C_{kt}. The feasibility of a particular merger (including its efficiency) is publicly observed by all firms. In each period, the firms involved in a feasible and not-yet-approved merger decide whether or not to propose their merger to the antitrust authority. Bargaining is efficient so that the merger partners propose the merger if and only if it is in their joint interest to do so. Given a set of proposed mergers, the antitrust authority then decides which mergers to approve (if any). An approved merger is consummated immediately. Finally, at the end of each period, the firms play the pricing game, given current market structure. All firms as well as the antitrust authority discount payoffs with factor $\delta \leq 1$.

Following Nocke and Whinston (2010), we define a *myopically CS-maximizing merger policy* as an approval policy, where in each period, given the set of proposed mergers and current market structure, the antitrust authority approves a set of mergers that maximizes consumer surplus in the current period. The *most lenient myopically CS-maximizing merger policy* is a *myopically CS-maximizing merger policy* that approves the largest such set (i.e., including CS-neutral mergers). (As shown in Nocke and Whinston (2010) such a policy is well-defined.)

The following proposition shows that Nocke and Whinston (2010)'s result on the dynamic optimality of a myopic merger approval policy carries over to our multiproduct firm setting:

Proposition XIX. *Suppose the antitrust authority adopts the most lenient myopically CS-maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect Nash equilibrium for the firms. The equilibrium outcome maximizes discounted consumer surplus (indirect utility) for any realized sequence of feasible mergers. Moreover, for each such sequence, every subgame-perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.*

Proof. The result follows from Propositions XIV, XVI, XVII, and XVIII, which are the analogues of Corollary 1 and Proposition 1 and 2 in Nocke and Whinston (2010). See Nocke and Whinston (2010) for details. □

XVI Application: International Trade

In this section, we study the effect of trade liberalization on the inter- and intra-firm size distributions, average industry-level productivity, and welfare. Throughout, we assume that demand is either of the (nested) CES or MNL forms so that the type aggregation property (see Section 5) holds. We show that a (unilateral) trade liberalization will magnify relative market share differences between domestic firms, and leave the intra-firm distribution of market shares unchanged.[39] Different predictions obtain for both the inter- and intra-firm *sales* distributions in the MNL case. A unilateral trade liberalization shifts relative sales toward firms with high market shares and high average costs, and within each firm toward high-cost products. We also propose a new measure of firm-level productivity that allows us to deal with the fact that firms are heterogeneous in terms of scope, and product-level marginal costs and qualities. We find that a trade liberalization raises average industry-level productivity by reallocating market shares toward more productive firms. Finally, we provide sufficient conditions for a unilateral trade liberalization to raise (or lower) domestic welfare.

We state our results in Section XVI.1, and prove them in Section XVI.2.

XVI.1 Trade Analysis

As argued in Section 3.3, a trade liberalization that improves the access of foreign firms to the domestic market can be thought of as an increase in the value of the outside option, H^0. By Proposition 4, the increase in the value of H^0 in turn leads to an increase in the equilibrium value of the aggregator, H, thereby benefiting consumers but hurting domestic firms. We now study the impact of the increase in H^0 on the inter- and intra-firm size distributions, average industry-level productivity, and welfare.

Inter-firm size distribution. As we show in Section XVI.2, the ratio of market shares between domestic firm f and a smaller domestic firm g of type $T^g < T^f$, $S(T^f/H)/S(T^g/H)$, is increasing in H. That is, a trade liberalization leads to a smaller fractional decrease in the market share of a larger than a smaller domestic firm, and thereby magnifies relative size differences. To put this result into perspective, consider the case of monopolistic competition, where firms take H as given. As demand is inversely proportional to H, a trade liberalization would, in that case, not affect prices and would thus have no impact on the market share ratio of two domestic firms. In contrast, under oligopolistic competition, firms reduce their markups in response to the increased competition from abroad. But a small oligopolistic firm (small T) will not reduce its markup much as it cannot affect H much; as a result, its fractional reduction in market share is almost as large as the fractional increase in H, and therefore larger than that of a large firm (large T).

Recall from Section 5 that firm f's market share $S(T^f/H)$ is measured in value (i.e., sales) in the case of CES demand, and in volume (i.e., output) in the case of MNL demand. As sales data are often more readily available than output data, in the following we also derive the effect of a trade liberalization on the domestic sales distribution in the MNL case. In that case, firm f's sales share depends not only on f's type T^f (and the aggregator H) but also on its (output-weighted) average cost $\bar{c}^f$:

$$\text{Sales}^f = \frac{1}{H}\left(\sum_{j\in f} p_j \exp\frac{a_j - p_j}{\lambda}\right) = \frac{T^f}{H}\,\mathrm{e}^{-\mu^f}\left(\lambda\mu^f + \bar{c}^f\right),$$

[39] Recall that market shares are defined in value under CES demand and in volume under MNL demand.

where

$$\bar{c}^f \equiv \sum_{k\in f}\left(\frac{e^{\frac{a_k-c_k}{\lambda}-\mu^f}}{\sum_{j\in f} e^{\frac{a_j-c_j}{\lambda}-\mu^f}}\right)c_k = \sum_{k\in f}\left(\frac{e^{\frac{a_k-c_k}{\lambda}}}{\sum_{j\in f} e^{\frac{a_j-c_j}{\lambda}}}\right)c_k.$$

As a result, the effect of a trade liberalization on the domestic inter-firm sales distribution is more subtle than in the CES case. As shown in Section XVI.2, there exists a function ϕ, decreasing in both arguments, such that a trade liberalization shifts sales from firm g toward firm f if and only if $\phi(T^f,\bar{c}^f) < \phi(T^g,\bar{c}^g)$. In particular, if (i) $T^f > T^g$ and $\bar{c}^f = \bar{c}^g$ or (ii) $T^f = T^g$ and $\bar{c}^f > \bar{c}^g$, firm f has larger sales than firm g, and a trade liberalization will magnify this size difference. But note that in case (ii) more sales are shifted toward the firm with the higher average cost.

Intra-firm size distribution. In Section XVI.2, we show that the ratio of market shares between any two products $j \in f$ and $k \in f$ offered by the same domestic firm f, s_j/s_k, is independent of the equilibrium value of the aggregator H. That is, the intra-firm distribution of market shares is invariant to the degree of international trade integration. This follows from two observations. First, the common ι-markup property implies a common relative markup under CES demand and a common absolute markup under MNL demand. While different firms reduce their markups to different degrees in response to the trade liberalization, each firm reduces all its prices by the same fraction (CES) or the same absolute amount (MNL). Second, all products have the same (constant) price elasticity of revenue (CES) or the same (constant) price semi-elasticity of demand (MNL).

Under MNL demand, a different picture emerges for the intra-firm sales distribution. As each firm reduces all its prices by the same absolute amount, a trade liberalization induces a shift in relative sales toward products with higher marginal costs: If $c_j > c_k$, then $(p_j s_j)/(p_k s_k)$ increases with an increase in H.

Firm-level productivity. The effects of a trade liberalization on industry-level productivity is a key question in the field of international trade. To address this question, we first define productivity at the firm level. As marginal costs and qualities are allowed to vary across products within the same firm and as firms may differ in the number of products they offer, such a firm-level definition is not obvious. Below, we provide two arguments suggesting that a monotonic transform of the firm's type T^f, $\varphi^f \equiv \varphi(T^f)$ for some strictly increasing function φ, is the theoretically correct measure of productivity.

The composite commodity approach. Suppose the representative consumer has CES preferences. Define the firm-level composite commodity $Q^f \equiv \left(\sum_{j\in f} a_j^{\frac{1}{\sigma}} q_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$ and the associated price index $P^f \equiv \left(\sum_{j\in f} a_j p_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$ for every firm f. It is well known that the demand system can then equivalently be derived through two-stage budgeting, where the consumer first decides on $(Q^f)_{f\in\mathcal{F}}$, and then on $(q_j)_{j\in f}$ for every f. For this composite commodity approach, firm f's productivity is equal to the inverse of the firm's (constant) unit cost of producing the composite commodity Q^f, which can be computed as $(T^f)^{\frac{1}{\sigma-1}}$. Equivalently, it is equal to the ratio of firm f's sales over costs, with sales being deflated by the firm-level price index P^f. Unfortunately, such composite commodities are unavailable for the MNL case.

The indirect utility approach. Suppose the consumer makes choices in two steps. He first chooses a firm, and then a product in that firm's "nest." After having chosen firm f, the consumer observes a vector of i.i.d. Gumbel taste shocks and chooses the product that delivers the largest indirect utility. The inclusive value of nest f is then given by $V^f \equiv \log \sum_{j \in f} h_j(p_j)$. Suppose firm f is tasked to deliver the inclusive value in a profit-maximizing way (i.e., conditional on the consumer choosing its products' nest). It is straightforward to show that the resulting prices satisfy the common ι-markup property, and that a firm with a higher T^f delivers any inclusive value V^f in a more efficient way.

Industry-level productivity. We define domestic industry-level productivity Φ as the market-share-weighted average firm-level productivity:

$$\Phi \equiv \sum_{f \in \mathcal{F}} \left(\frac{s^f}{\sum_{g \in \mathcal{F}} s^g} \right) \varphi^f.$$

The reason for using market shares as weights is that, under both CES and MNL demands, these market shares are equal to firm-level choice probabilities in the discrete/continuous choice model.

We now consider the effect of a (unilateral) trade liberalization on Φ. An immediate observation is that firm-level productivities remain unchanged.[40] However, as a trade liberalization increases the relative market shares of high-productivity firms, it induces an increase in the average industry-level productivity Φ. This is shown formally in Section XVI.2.

Welfare effects. By increasing the equilibrium value of the aggregator H, a unilateral trade liberalization benefits domestic consumers. As shown above, it also raises the average industry-level productivity of domestic firms. The underlying shift in market shares from less productive to more productive domestic firms reduces the relative markup distortions discussed in Section 3.3. The downside of a unilateral trade liberalization is that it increases the market share of foreign firms (the outside option).

The overall effect on domestic welfare, defined as the sum of consumer surplus and the profit of domestic firms, is identical to the external effect of a CS-increasing merger. From our analysis in Section XV.1, we obtain the following. First, if $\sigma < \bar{\sigma}$ under CES demand, then a unilateral trade liberalization always increases domestic welfare. Second, under MNL

[40] Recall that, in the CES case, productivity can be measured by the ratio of firm-level sales over costs, with sales being deflated by the firm-level CES price index. Without access to the ideal firm-level price index, it would appear that a trade-liberalization does affect firm-level productivity. The same applies to the MNL case, for which ideal firm-level price indices do not even exist theoretically.

demand or if $\sigma > \bar{\sigma}$ under CES demand, a small unilateral trade liberalization is more likely to have a positive domestic welfare effect if (i) the joint market share of domestic firms is smaller, and if (ii) the domestic industry is less concentrated. In contrast, under monopolistic competition with our class of demand systems (see Proposition XXV) or linear demand (see Mayer, Melitz, and Ottaviano, 2014), a unilateral trade liberalization would have an unambiguously positive effect on domestic welfare (for a fixed set of firms).

XVI.2 Formal Statements and Proofs

Inter-firm size distribution

Proposition XX. *Suppose demand is either of the CES or MNL form. Then, for $T^f > T^g$, the ratio $S(T^f/H)/S(T^g/H)$ is increasing in H. That is, a trade liberalization leads to a smaller fractional decrease in the market share of a larger than a smaller firm.*

Proof. We have

$$\frac{d}{dH}\left(\frac{S\left(\frac{T^f}{H}\right)}{S\left(\frac{T^g}{H}\right)}\right) > 0$$

if and only if

$$\frac{\frac{T^f}{H}S'\left(\frac{T^f}{H}\right)}{S\left(\frac{T^f}{H}\right)} < \frac{\frac{T^g}{H}S'\left(\frac{T^g}{H}\right)}{S\left(\frac{T^g}{H}\right)}.$$

By Lemma XXV, $\epsilon'(x) < 0$ for all x, where $\epsilon(x) \equiv xS'(x)/S(s)$. Hence, the inequality holds if and only if $T^f > T^g$. □

Proposition XXI. *Suppose demand is of the MNL form. Then, for $T^f > T^g$, the sales ratio between firms f and g is increasing in H if and only if $\phi(s^f, \bar{c}^f) < \phi(s^g, \bar{c}^g)$, where $s^i = S(T^i/H)$ and*

$$\bar{c}^i \equiv \sum_{k\in i} \frac{e^{\frac{a_k - c_k}{\lambda}}}{\sum_{j\in i} e^{\frac{a_j - c_j}{\lambda}}} c_k$$

are, respectively, the market share (in volume) and the (output-weighted) average marginal cost of firm $i \in \{f, g\}$, and

$$\phi(s, \bar{c}) \equiv \frac{1-s}{1-s+s^2}\left(1 - s + \frac{s}{1 + \frac{\bar{c}(1-s)}{\lambda}}\right)$$

is decreasing in s and $\bar{c}$.

Proof. Firm f's sales can be written as

$$\text{Sales}^f = \frac{1}{H}\left(\sum_{j\in f} p_j \exp\frac{a_j - p_j}{\lambda}\right),$$

$$= \frac{1}{H}\left(\sum_{j\in f}(\lambda\mu^f + c_j)\,\mathrm{e}^{-\mu^f}\exp\frac{a_j - c_j}{\lambda}\right),$$
$$= \frac{T^f}{H}\,\mathrm{e}^{-\mu^f}\left(\lambda\mu^f + \bar{c}^f\right).$$

The fact that $s^f = \frac{T^f}{H}\,\mathrm{e}^{-\mu^f}$ and $\mu^f = \frac{1}{1-s^f}$ allows us to rewrite firm f's sales as follows:

$$\text{Sales}^f = s^f\left(\frac{\lambda}{1-s^f} + \bar{c}^f\right).$$

The logarithmic derivative of sales with respect to H is given by:

$$\frac{d\log \text{Sales}^f}{dH} = \frac{ds^f}{dH}\left(\frac{1}{s^f} + \frac{1}{(1-s^f) + \frac{\bar{c}^f(1-s^f)^2}{\lambda}}\right),$$
$$= -\frac{T^f}{H^2}S'\left(\frac{T^f}{H}\right)\left(\frac{1}{s^f} + \frac{1}{(1-s^f) + \frac{\bar{c}^f(1-s^f)^2}{\lambda}}\right),$$
$$= -\frac{1}{H}\varepsilon\left(\frac{T^f}{H}\right)s^f\left(\frac{1}{s^f} + \frac{1}{(1-s^f) + \frac{\bar{c}^f(1-s^f)^2}{\lambda}}\right),$$

where ε is the elasticity of S. Recall that (see equation (xlvi))

$$\varepsilon = \frac{(1-S)^2}{1-S+S^2}.$$

It follows that

$$\frac{d\log \text{Sales}^f}{dH} = -\frac{1}{H}\underbrace{\frac{1-s^f}{1-s^f+s^{f2}}\left(1 - s^f + \frac{s^f}{1+\frac{\bar{c}^f(1-s^f)}{\lambda}}\right)}_{\equiv\phi(s^f,\bar{c}^f)}$$

and hence,

$$\frac{d\log(\text{Sales}^f/\text{Sales}^g)}{dH} = \frac{1}{H}\left(\phi(s^g,\bar{c}^g) - \phi(s^f,\bar{c}^f)\right).$$

It can be verified that ϕ is decreasing in both arguments. □

Intra-firm size distribution

Proposition XXII. *Suppose demand is either of the CES or MNL form. Then, for $j,k\in f$, the market share ratio s_j/s_k is independent of H. That is, a trade liberalization leads to the same fractional decrease in the market share of all products offered by the same firm.*

Proof. Consider first the case of CES demand. The ratio of market shares (in value) between any two products $j, k \in f$ is given by

$$\frac{s_j}{s_k} = \frac{a_j}{a_k}\left(\frac{p_j}{p_k}\right)^{1-\sigma} = \frac{a_j}{a_k}\left(\frac{\frac{c_j}{1-\mu^f/\sigma}}{\frac{c_k}{1-\mu^f/\sigma}}\right)^{1-\sigma} = \frac{a_j}{a_k}\left(\frac{c_j}{c_k}\right)^{1-\sigma}.$$

Hence, the market share ratio is independent of H.

Consider now the case of MNL demand. The ratio of market shares (in volume) between any two products $j, k \in f$ is given by

$$\frac{s_j}{s_k} = \frac{\mathrm{e}^{\frac{a_j-p_j}{\lambda}}}{\mathrm{e}^{\frac{a_k-p_k}{\lambda}}} = \frac{\mathrm{e}^{\frac{a_j-c_j}{\lambda}-\mu^f}}{\mathrm{e}^{\frac{a_k-c_k}{\lambda}-\mu^f}} = \frac{\mathrm{e}^{\frac{a_j-c_j}{\lambda}}}{\mathrm{e}^{\frac{a_k-c_k}{\lambda}}},$$

which is independent of H. □

Proposition XXIII. *Suppose demand is of the MNL form. Then, for $j, k \in f$, with $c_j > c_k$, the sales ratio $(p_j s_j)/(p_k s_k)$ is increasing in H. That is, within each firm, a trade liberalization leads to a larger fractional increase in the sales of a product that is produced at higher marginal cost.*

Proof. The ratio of sales of any two products $j, k \in f$ is given by

$$\frac{p_j s_j}{p_k s_k} = \frac{c_j + \lambda\mu^f}{c_k + \lambda\mu^f}\frac{\mathrm{e}^{\frac{a_j-c_j}{\lambda}}}{\mathrm{e}^{\frac{a_k-c_k}{\lambda}}}.$$

As an increase in H induces a decrease in the markup μ^f, this ratio is increasing in H if and only if $c_j > c_k$. □

Productivity. We argue in Section XVI.1 that a monotone transformation of firm f's type provides a theoretically sound measure of that firm's productivity. We now prove this assertion formally.

The composite commodity approach. Assume that demand is of the CES form, and let $\alpha = (\sigma - 1)/\sigma$. The composite commodity produced by firm f has been defined as $Q^f = \left(\sum_{j\in f} a_j^{1-\alpha} q_j^{\alpha}\right)^{\frac{1}{\alpha}}$. Suppose that firm f has been tasked to produce a certain level Q^f of composite commodity in a cost-minimizing way. Then, firm f solves the following cost-minimization problem:

$$\min_{(q_j)_{j\in f}} \sum_{j\in f} c_j q_j \quad \text{s.t. } Q^f = \left(\sum_{j\in f} a_j^{1-\alpha} q_j^{\alpha}\right)^{\frac{1}{\alpha}}.$$

The first-order condition for product $i \in f$ is:

$$c_i - \Lambda (Q^f)^{1-\alpha} a_i^{1-\alpha} q_i^{\alpha-1} = 0,$$

where Λ is the Lagrange multiplier associated with the output constraint. Multiplying the first-order condition by q_i and adding up, we obtain: $\sum_{j\in f} c_j q_j = \Lambda Q^f$.

Moreover,

$$q_i = \left(\frac{\Lambda}{c_i}\right)^{\frac{1}{1-\alpha}} a_i Q^f.$$

Therefore,

$$(Q^f)^\alpha = \sum_{i\in f} a_i^{1-\alpha} q_i^\alpha = \sum_{i\in f} \left(\frac{\Lambda}{c_i}\right)^{\frac{\alpha}{1-\alpha}} a_i (Q^f)^\alpha.$$

It follows that

$$\Lambda^{1-\sigma} = \Lambda^{-\frac{\alpha}{1-\alpha}} = \sum_{i\in f} a_i c_i^{-\frac{\alpha}{1-\alpha}} = \sum_{i\in f} a_i c_i^{1-\sigma} = T^f.$$

Therefore, $\Lambda = (T^f)^{\frac{1}{1-\sigma}}$. Recall that $\Lambda = \frac{\sum_{j\in f} c_j q_j}{Q^f}$. This implies that firm f's implied production technology for the composite commodity has constant returns to scale, and that firm f's constant unit cost is equal to $(T^f)^{\frac{1}{1-\sigma}}$. Put differently, firm f's productivity for the composite commodity is $(T^f)^{\frac{1}{\sigma-1}}$.

The indirect utility approach. Firm f is tasked to deliver the inclusive value V^f in a profit-maximizing way. That is, firm f solves maximization problem

$$\max_{(p_j)_{j\in f}} \sum_{j\in f} (p_j - c_j) \frac{-h_j'}{\mathrm{e}^{V^f}} \quad \text{s.t. } \log \sum_{j\in f} h_j(p_j) = V^f.$$

It is straightforward to show that firm f's profile of prices must satisfy the constant ι-markup property: There exists μ^f such that $p_j - c_j = \lambda \mu^f$ (resp. $\sigma \frac{p_j - c_j}{p_j} = \mu^f$) in the MNL (resp. CES) case for every $j \in f$.

The optimal value of μ^f is pinned down by the inclusive-value constraint:

$$\log \sum_{j\in f} h_j(r_j(\mu^f)) = V^f.$$

This yields $\mu^f = \log T^f - V^f$ in the MNL case, and $\mu^f = \sigma\left(1 - \left(T^f \mathrm{e}^{-V^f}\right)^{\frac{1}{1-\sigma}}\right)$ in the CES case. Plugging this value of μ^f into the objective function, we find that firm f makes a profit of $\log T^f - V^f$ in the MNL case, and $(\sigma - 1)\left(1 - \left(T^f \mathrm{e}^{-V^f}\right)^{\frac{1}{1-\sigma}}\right)$ in the CES case. In both cases, a firm with a higher T^f delivers the inclusive value V^f in a more efficient way.

Next, we study the impact of trade liberalization on domestic industry-level productivity:

Proposition XXIV. *With CES or MNL demands, a trade liberalization raises the domestic industry-level productivity.*

Proof. Assume without loss of generality that $\mathcal{F} = \{1, \dots, n\}$, and that $T^1 \leq \dots \leq T^n$. Let $(s^f)_{1\leq f\leq n}$ (resp. $(s'^f)_{1\leq f\leq n}$) be the pre-trade liberalization (resp. post-trade liberalization) vector of market shares. Define also Φ and Φ' as the pre- and post-trade liberalization industry-level productivity, respectively. By Proposition XX, we have that $s'^f s^g \leq s^f s'^g$ whenever $f \leq g$.

For every $1 \leq f \leq n$, define $w^f = s^f / \sum_{g=1}^n s^g$ and $w'^f = s'^f / \sum_{g=1}^n s'^g$. We interpret $w \equiv (w^f)_{1\leq f\leq n}$ and $w' \equiv (w'^f)_{1\leq f\leq n}$ as a discrete probability distributions over $\{1, \dots, n\}$. We claim that w' first-order stochastically dominates w. To see this, let $F \in \{1, \dots, n\}$, and note that

$$\begin{aligned}
\sum_{f=1}^F w'^f \leq \sum_{f=1}^F w^f &\iff \frac{\sum_{f=1}^F s'^f}{\sum_{f=1}^n s'^f} \leq \frac{\sum_{f=1}^F s^f}{\sum_{f=1}^n s^f}, \\
&\iff \sum_{f=1}^F s'^f \sum_{g=F+1}^n s^g \leq \sum_{f=1}^F s^f \sum_{g=F+1}^n s'^g, \\
&\iff \sum_{f=1}^F \sum_{g=F+1}^n \underbrace{\left(s'^f s^g - s^f s'^g\right)}_{\leq 0} \leq 0,
\end{aligned}$$

which holds true by Proposition XX. This confirms that w' first-order stochastically dominates w.

Since the functions $\varphi(\cdot)$ and $f \mapsto T^f$ are increasing, it follows that

$$\Phi' = \sum_{f=1}^n w'^f \varphi(T^f) \geq \sum_{f=1}^n w^f \varphi(T^f) = \Phi. \qquad \square$$

Welfare

Proposition XXV. *Suppose that the demand system $(h_j)_{j\in\mathcal{N}}$ satisfies Assumption 1. Then, under monopolistic competition, a unilateral trade liberalization raises domestic welfare.*

Proof. Recall that, under monopolistic competition, every firm sets a ι-markup of 1. Domestic social welfare is therefore given by:

$$\begin{aligned}
W(H^0) &= \log\left(\sum_{j\in\mathcal{N}} h_j(r_j(1)) + H^0\right) + \sum_{j\in\mathcal{N}} (r_j(1) - c_j) \frac{-h_j'(r_j(1))}{\sum_{k\in\mathcal{N}} h_k(r_k(1)) + H^0}, \\
&= \log\left(\sum_{j\in\mathcal{N}} h_j(r_j(1)) + H^0\right) + \frac{\sum_{j\in\mathcal{N}} \gamma_j(r_j(1))}{\sum_{k\in\mathcal{N}} h_k(r_k(1)) + H^0}.
\end{aligned}$$

Differentiating W, we obtain:

$$W'(H^0) = \frac{1}{\sum_{j\in\mathcal{N}} h_j(r_j(1)) + H^0} \left(1 - \frac{\sum_{j\in\mathcal{N}} \gamma_j(r_j(1))}{\sum_{k\in\mathcal{N}} h_k(r_k(1)) + H^0}\right),$$

which is strictly positive, by log-convexity of h_j. □

XVII Table of Symbols and Notations

Market-level notations	
H	Aggregator, sufficient statistic for consumer surplus
H^0	Outside option
$\Gamma(H)$	Aggregate fitting-in function
$\Omega(H)$	$\Gamma(H)/H$, aggregate share function
$\mathcal{N}$	Set of products
$\mathcal{F}$	Set of firms
Firm-level notations	
μ^f	Firm f's ι-markup
$m^f(H)$	Firm f's fitting-in function
$\bar{\mu}^f$	$\max_{k\in f}\bar{\mu}_k$, the highest ι-markup that firm f can sustain
ω^f	$(\mu^f-1)/\mu^f$
T^f	Firm f's type (CES / MNL demands)
Product-level notations	
$\mathcal{H}$	The set of $\mathcal{C}^3$, strictly decreasing and log-convex functions
$\mathcal{H}^\iota$	The set of functions in $\mathcal{H}$ that satisfy Assumption 1
h_k	Exponential of indirect subutility derived from product k
$-h'_k/h_k$	Conditional demand for product k
$h_k/(H^0+\sum_{j\in\mathcal{N}}h_j)$	Choice probability for product k
ι_k	$p_k h''_k(p_k)/(-h'_k(p_k))$, elasticity of monopolistic competition demand
$\bar{\mu}_k$	$\lim_{p_k\to\infty}\iota_k(p_k)$, the highest ι-markup that product k can sustain
γ_k	h'^2_k/h''_k
ρ_k	h_k/γ_k
θ_k	h'_k/γ'_k
χ_k	$(\iota_k-1)/(\iota_k)$
$\nu_k(p_k)$	$\iota_k(p_k)(p_k-c_k)/p_k$, ι-markup on product k
$r_k(\mu^f)$	$\nu_k^{-1}(\mu^f)$, pricing function
p_k^{mc}	$r_k(1)$, product k's price under monopolistic competition
$\underline{p}_k$	$\inf\{p_k>0:\ \iota_k(p_k)>1\}$

References

Anderson, S. P., A. de Palma, and J.-F. Thisse (1992): *Discrete Choice Theory of Product Differentiation*. Cambridge, MA: MIT Press.

Anderson, S. P., N. Erkal, and D. Piccinin (2013): "Aggregate Oligopoly Games with Entry," CEPR Discussion Paper 9511, Center for Economic Policy Research.

Atkeson, A., and A. Burstein (2008): "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, 98(5), 1998–2031.

Bernard, A. B., S. J. Redding, and P. K. Schott (2010): "Multiple-Product Firms and Product Switching," *American Economic Review*, 100(1), 70–97.

——— (2011): "Multiproduct Firms and Trade Liberalization," *The Quarterly Journal of Economics*, 126(3), 1271–1318.

Bertoletti, P., and F. Etro (2017): "Monopolistic Competition when Income Matters," *The Economic Journal*, 127(603), 1217–1243.

Breinlich, H., V. Nocke, and N. Schutz (2015): "Merger Policy in a Quantitative Model of International Trade," CEPR Discussion Paper 10851, Center for Economic Policy Research.

Deneckere, R., and C. Davidson (1985): "Incentives to Form Coalitions with Bertrand Competition," *RAND Journal of Economics*, 16(4), 473–486.

Dhingra, S. (2013): "Trading Away Wide Brands for Cheap Brands," *American Economic Review*, 103(6), 2554–84.

Eckel, C., L. Iacovone, B. Javorcik, and J. P. Neary (2015): "Multi-Product Firms at Home and Away: Cost- versus Quality-Based Competence," *Journal of International Economics*, 95(2), 216 – 232.

Eckel, C., and J. P. Neary (2010): "Multi-Product Firms and Flexible Manufacturing in the Global Economy," *Review of Economic Studies*, 77(1), 188–217.

Edmond, C., V. Midrigan, and D. Y. Xu (2015): "Competition, Markups, and the Gains from International Trade," *American Economic Review*, 105(10), 3183–3221.

Farrell, J., and C. Shapiro (1990): "Horizontal Mergers: An Equilibrium Analysis," *American Economic Review*, 80(1), 107–26.

Hanemann, W. M. (1984): "Discrete-Continuous Models of Consumer Demand," *Econometrica*, 52(3), 541–61.

Hottman, C. J., S. J. Redding, and D. E. Weinstein (2016): "Quantifying the Sources of Firm Heterogeneity," *The Quarterly Journal of Economics*, 131(3), 1291–1364.

Mayer, T., M. J. Melitz, and G. I. P. Ottaviano (2014): "Market Size, Competition, and the Product Mix of Exporters," *American Economic Review*, 104(2), 495–536.

Monderer, D., and L. S. Shapley (1996): "Potential Games," *Games and Economic Behavior*, 14(1), 124 – 143.

Neary, J. P. (2003): "Presidential Address: Globalization and Market Structure," *Journal of the European Economic Association*, 1(2-3), 245–271.

——— (2016): "International Trade in General Oligopolistic Equilibrium," *Review of International Economics*, 24(4), 669–698.

Nocke, V., and N. Schutz (2016): "Multiproduct-Firm Pricing Games and Transformed Potentials," Unpublished manuscript.

——— (2017): "Quasi-linear integrability," *Journal of Economic Theory*, 169(C), 603–628.

Nocke, V., and M. D. Whinston (2010): "Dynamic Merger Review," *Journal of Political Economy*, 118(6), 1201 – 1251.

Nocke, V., and S. Yeaple (2014): "Globalization and Multiproduct Firms," *International Economic Review*, 55, 993–1018.

Novshek, W., and H. Sonnenschein (1979): "Marginal Consumers and Neoclassical Demand Theory," *Journal of Political Economy*, 87(6), 1368–76.

Qiu, L., and W. Zhou (2013): "Multiproduct Firms and Scope Adjustment in Globalization," *Journal of International Economics*, 91(1), 142–153.

Slade, M. E. (1994): "What Does an Oligopoly Maximize?," *The Journal of Industrial Economics*, 42(1), 45–61.

Williamson, O. E. (1968): "Economies as an Antitrust Defense: The Welfare Tradeoffs," *American Economic Review*, 58(1), 18–36.