Quasi-Linear Integrability: Addendum*

Volker Nocke† Nicolas Schutz‡

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We have recently become aware of existing results in the mathematics literature on the subgradients of convex functions (Rockafellar, 1970). These results have been used in the mechanism design literature by Rochet (1987), Jehiel, Moldovanu, and Stacchetti (1999), Krishna and Maenner (2001), and others. They can also be used to prove the equivalence of parts (ii), i.e., the cyclical monotonicity of the demand function $D(\cdot)$, and (iii), i.e., $D(\cdot)$ being a conservative vector field satisfying the law of demand, in Theorem 1 of Nocke and Schutz (2017).

Specifically, the proof of the equivalence of (ii) and (iii), Lemma 11 in Nocke and Schutz (2017), can be shortened as follows. First, that (iii) implies (ii) follows directly from Theorem 24.8 in Rockafellar (1970). Second, to show that (ii) implies (iii), we can use again Theorem 24.8 in Rockafellar (1970) to obtain the existence of a convex function $V(\cdot)$ such that for every price vector p, $-D(p)$ belongs to the subgradient of $V(\cdot)$ at price vector p. The result that $-D(\cdot)$ is, in fact, the gradient of $V(\cdot)$, can then be established by applying Theorem 1 in Krishna and Maenner (2001): For any two distinct price vectors p and p',

$$V(p') - V(p) = -\int_0^1 D(tp' + (1-t)p) \cdot (p' - p)dt.$$

The continuity of $D(\cdot)$ immediately implies that

$$\frac{V(p') - V(p) + (p' - p) \cdot D(p)}{||p' - p||} \underset{p' \to p}{\longrightarrow} 0.$$

Hence, $V(\cdot)$ is differentiable, and its gradient is indeed $-D(\cdot)$. This establishes that (ii) implies that there exists a convex function $V(\cdot)$ such that for every price vector p, $-D(p)$ is the gradient of $V(\cdot)$, which in turn, by Lemmas 13 and 14 in Nocke and Schutz (2017), is equivalent to (iii).

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†UCLA and CEPR. Email: volker.nocke@gmail.com.
‡University of Mannheim. Email: schutz@uni-mannheim.de.

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