Online Appendix to: An Aggregative Games Approach to Merger Analysis in Multiproduct-Firm Oligopoly

Volker Nocke* Nicolas Schutz†

February 4, 2024

C Technical Results on Fitting-In Functions

The following results are proved in Nocke and Schutz (2018):

Lemma 5. *The following holds for every $\alpha \in (0, 1]$:*

(a) For every $x > 0$,

$$S'(x) = \frac{1}{x} \frac{S(x)(1 - S(x))(1 - \alpha S(x))}{1 - S(x) + \alpha S(x)^2}. \tag{28}$$

(b) The elasticity of S, $\varepsilon(x) = xS'(x)/S(x)$, is strictly decreasing in x.

(c) S is strictly concave.

Proof. See Section XIII.3 in the Online Appendix to Nocke and Schutz (2018). □

We also require the following lemma:

Lemma 6. *The continuous extension of S to $\mathbb{R}_+$ is $\mathcal{C}^2$. Moreover, $S(0) = 0$,*

$$S'(0) = \begin{cases} \alpha^{\frac{\alpha}{1-\alpha}} & \text{under NCES demand,} \\ e^{-1} & \text{under NMNL demand,} \end{cases}$$

and $S''(0) = -2\alpha S'(0)^2$.

The inverse function $\Theta \equiv S^{-1}$ is $\mathcal{C}^2$ on $[0, 1)$. Moreover, $\Theta(0) = 0$, $\Theta'(0) = 1/S'(0)$, and $\Theta''(0) = 2\alpha/S'(0)$.

*Department of Economics and MaCCI, University of Mannheim; also affiliated with CEPR. Email: volker.nocke@gmail.com.

†Department of Economics and MaCCI, University of Mannheim; also affiliated with CEPR. Email: schutz@uni-mannheim.de.

Proof. We start by computing $\lim_{x\downarrow 0}\frac{S(x)}{x}$. In the NMNL case,

$$\frac{S(x)}{x} = \mathrm{e}^{-m(x)} = \exp\left(\frac{-1}{1-S(x)}\right) \underset{x\downarrow 0}{\longrightarrow} \mathrm{e}^{-1}.$$

In the NCES case,

$$\frac{S(x)}{x} = (1-(1-\alpha)m(x))^{\frac{\alpha}{1-\alpha}} = \left(1-\frac{1-\alpha}{1-\alpha S(x)}\right)^{\frac{\alpha}{1-\alpha}} \underset{x\downarrow 0}{\longrightarrow} \alpha^{\frac{\alpha}{1-\alpha}}.$$

Differentiating equation (28), we obtain

$$S''(x) = -\left(\frac{S(x)}{x}\right)^2 \frac{\alpha(2-S(x))(1-S(x))(1-\alpha S(x))}{(1-S(x)+\alpha S(x)^2)^3}.$$

Taking limits yields $S''(0)$.

As S is $\mathcal{C}^2$ with strictly positive derivative on $\mathbb{R}_+$, that function establishes a $\mathcal{C}^2$-diffeomorphism from $\mathbb{R}_+$ to

$$\left[S(0), \lim_{x\to\infty} S(x)\right) = [0,1).$$

It follows that Θ is $\mathcal{C}^2$. Moreover,

$$\Theta'(s) = \frac{1}{S'\circ S^{-1}(s)},$$
$$\Theta''(s) = -\frac{S''\circ S^{-1}(s)}{(S'\circ S^{-1}(s))^3}.$$

Hence,

$$\Theta'(0) = \frac{1}{S'(0)},$$
$$\Theta''(0) = -\frac{1}{S'(0)}\frac{S''(0)}{S'(0)^2} = \frac{2\alpha}{S'(0)}.$$

□

D Consumer Surplus Effects: Static Analysis

Proof of Proposition 3. Recall that $\varepsilon(\cdot)$ is the elasticity of S (see Lemma 5) and that the cutoff type solves the equation:

$$S\left(\frac{\hat{T}^M}{H^*}\right) = \sum_{f\in\mathcal{M}} S\left(\frac{T^f}{H^*}\right).$$

Totally differentiating this equation, we obtain:

$$\begin{aligned}
S'\left(\frac{\hat{T}^M}{H^*}\right)\frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*}S'\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f\in\mathcal{M}} \frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right) S\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f\in\mathcal{M}} \varepsilon\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right) \sum_{f\in\mathcal{M}} S\left(\frac{T^f}{H^*}\right) - \sum_{f\in\mathcal{M}} \varepsilon\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right), \\
&= \sum_{f\in\mathcal{M}} \left(\varepsilon\left(\frac{\hat{T}^M}{H^*}\right) - \varepsilon\left(\frac{T^f}{H^*}\right)\right) S\left(\frac{T^f}{H^*}\right), \\
&< 0,
\end{aligned}$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma 5 and from the fact that $\hat{T}^M > T^f$ for every $f \in \mathcal{M}$. □

Proof of Proposition 4. Note that

$$\frac{\hat{T}^M}{T^f + T^g} = \frac{S^{-1}\left(S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right)}{\frac{T^f}{H^*} + \frac{T^g}{H^*}} = \xi\left(\frac{T^f}{H^*}, \frac{T^g}{H^*}\right),$$

where

$$\xi(x, y) \equiv \frac{S^{-1}\left(S(x) + S(y)\right)}{x + y}, \quad \forall x, y > 0.$$

Proving the proposition therefore boils down to showing that $\partial\xi/\partial x > 0$ and $\partial\xi/\partial y > 0$. By symmetry, this is equivalent to proving that $\partial\xi/\partial x > 0$, which we undertake next.

Differentiating ξ with respect to x, we obtain:

$$\frac{\partial\xi}{\partial x} = \frac{S^{-1}(S(x) + S(y))}{(x + y)^2}\left(\underbrace{\frac{(x + y) \times S'(x)}{S^{-1}(S(x) + S(y)) \times S' \circ S^{-1}(S(x) + S(y))}}_{\equiv\psi(x,y)} - 1\right).$$

Let $z = S^{-1}(S(x) + S(y))$. By definition, $S(z) = S(x) + S(y)$. Moreover, by subadditivity of S, $z > x + y$. Assume first that $x \leq y$. Note that

$$\begin{aligned}
\psi(x, y) &= \frac{(x + y)S'(x)}{zS'(z)}, \\
&= \frac{(x + y)S'(x)/(S(x) + S(y))}{zS'(z)/S(z)},
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{xS'(x)}{S(x)}\frac{S(x)}{S(x)+S(y)} + \frac{yS'(x)}{S(y)}\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \\
&\geq \frac{\frac{xS'(x)}{S(x)}\frac{S(x)}{S(x)+S(y)} + \frac{yS'(y)}{S(y)}\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \text{ by concavity of } S \text{ (see Lemma 5)}, \\
&= \frac{\varepsilon(x)\frac{S(x)}{S(x)+S(y)} + \varepsilon(y)\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \\
&> \frac{\varepsilon(z)\frac{S(x)}{S(x)+S(y)} + \varepsilon(z)\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \text{ as } \varepsilon \text{ is decreasing (see Lemma 5)}, \\
&= 1.
\end{aligned}
$$

Therefore, $\partial\xi/\partial x > 0$ whenever $x \leq y$.

Next, assume for a contradiction that $\psi(x, y) \leq 1$ for some $x > y$. Take the smallest such x. By continuity, this x exists, and satisfies $x > y$ (as shown in the first step of the proof) and $\psi(x, y) = 1$. Note that

$$
\begin{aligned}
\frac{\partial\psi}{\partial x} &= \frac{1}{(zS'(z))^2}\left(\left(S'(x) + (x+y)S''(x)\right)zS'(z) - (x+y)S'(x)\left(S'(x) + S'(x)\frac{zS''(z)}{S'(z)}\right)\right), \\
&= \frac{1}{(zS'(z))^2}\left((x+y)S''(x)zS'(z) - (x+y)(S'(x))^2\frac{zS''(z)}{S'(z)}\right), \text{ as } \psi(x,y) = 1, \\
&= \frac{(x+y)z}{(zS'(z))^2}\left(S''(x)S'(z) - (S'(x))^2\frac{S''(z)}{S'(z)}\right), \\
&= \frac{(x+y)z(S'(x))^2S'(z)}{(zS'(z))^2}\left(\frac{S''(x)}{(S'(x))^2} - \frac{S''(z)}{(S'(z))^2}\right).
\end{aligned}
$$

Next, we argue that $S''(\cdot)/(S'(\cdot))^2$ is decreasing. Recall from Lemma 5 that

$$
S'(x) = \frac{1}{x}\frac{S(x)(1 - S(x))(1 - \alpha S(x))}{1 - S(x) + \alpha S(x)^2}.
$$

It follows that

$$
S''(x) = -\frac{\alpha(2 - S(x))(1 - S(x))(1 - \alpha S(x))S(x)^2}{x^2\left(1 - S(x)1 + \alpha S(x)^2\right)^3}.
$$

Hence,

$$
\frac{S''(x)}{(S'(x))^2} = -\frac{\alpha(2 - S(x))}{(1 - S(x))(1 - \alpha S(x))\left(1 - S(x)1 + \alpha S(x)^2\right)}.
$$

As $S(\cdot)$ is strictly increasing, the above expression is strictly decreasing in x if and only if

$$
\varphi(s) = \frac{\alpha(2 - s)}{(1 - s)(1 - \alpha s)\left(1 - s1 + \alpha s^2\right)}
$$

is strictly increasing in s. Routine calculations show that $\varphi'(s) > 0$ for every $s \in (0, 1)$ and

$\alpha \in (0,1]$. Therefore, $\partial\psi(x,y)/\partial x > 0$. It follows that $\psi(x',y) < 1$ in a small neighborhood to the left of x. This contradicts the definition of x. We can conclude that ξ is increasing in both of its arguments, which proves the proposition.[1] $\square$

E External Effects

We begin by deriving the formula for $\eta(H)$ (equation (12)):

Lemma 7. *$\eta(H)$ is given by:*

$$\eta(H) = -1 + \sum_{f\in\mathcal{O}} \phi(s^f, \alpha),$$

where $s^f = S(T^f/H)$, and

$$\phi(s,\alpha) = \frac{\alpha s(1-s)}{(1-\alpha s)(1-s+\alpha s^2)}, \quad \forall s \in (0,1),\ \forall \alpha \in (0,1].$$

Proof. This follows from the definition of η and from the fact that

$$\begin{aligned}
xm'(x) &= x\alpha\frac{S'(x)}{(1-\alpha S(x))^2}, \text{ as } m(x) = \frac{1}{1-\alpha S(x)},\\
&= \frac{\alpha}{(1-\alpha S(x))^2}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}, \text{ by Lemma 5,}\\
&= \frac{\alpha S(x)(1-S(x))}{(1-\alpha S(x))(1-S(x)+\alpha S(x)^2)},\\
&= \phi(S(x),\alpha).
\end{aligned}$$

$\square$

Next, we put on record the following facts about the function ϕ:

Lemma 8. *Let $\hat{\alpha} = \frac{1}{2} + \frac{\sqrt{33}}{18} \simeq 0.82$. The function ϕ has the following properties:*

(a) For every $s \in (0,1)$, $\phi(s,\cdot)$ is strictly increasing.

(b) If $\alpha \leq \hat{\alpha}$, then $\phi(s,\alpha) \leq s$ for every $s \in (0,1)$.

Moreover, if $\alpha > \hat{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0,1]$ and $\hat{s}(\alpha) \in (1/4,1)$ such that:

(c) $\phi(\cdot,\alpha)$ is strictly increasing on $(0, s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha), 1)$.

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[1] To see why $\hat{T}^M - (T^f + T^g) > \hat{T}^{M'} - (T^{f'} + T^{g'})$ (as mentioned in footnote 21 in the article), note that

$$\frac{\hat{T}^M - (T^f + T^g)}{T^{f'} + T^{g'}} > \frac{\hat{T}^M - (T^f + T^g)}{T^f + T^g} > \frac{\hat{T}^{M'} - (T^{f'} + T^{g'})}{T^{f'} + T^{g'}},$$

where the first inequality follows from the fact that $T^f + T^g > T^{f'} + T^{g'}$ and the second inequality follows from the first part of the proposition.

(d) $\phi(\cdot,\alpha)$ is strictly convex on $(0,\hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha),1)$.

Proof. We prove the lemma (analytically) using Mathematica. Mathematica files are available upon request. □

We are now in a position to prove the proposition:

Proof of Proposition 7. We begin by proving the first part of the proposition. If $\alpha \le \hat{\alpha}$, then, by Lemma 8, $\phi(x,\alpha) \le x$ for every $x \in (0,1)$. As outsiders' market shares add up to strictly less than 1, Lemma 7 immediately implies that any infinitesimal CS-decreasing merger has a negative external effect. Hence, any (not necessarily infinitesimal) CS-decreasing merger has a negative external effect.

Next, suppose $\alpha > \hat{\alpha}$, and define

$$\mathcal{S} = \bigcup_{n\ge 1} \mathcal{S}^n, \text{ where } \mathcal{S}^n = \{s \in [0,1]^n : \sum_{i=1}^{n} s_i \le 1\} \ \forall n \ge 1,$$
$$\bar{\mathcal{S}} = \bigcup_{n\ge 1} \bar{\mathcal{S}}^n, \text{ where } \bar{\mathcal{S}}^n = \{s \in [0,1]^n : \sum_{i=1}^{n} s_i = 1\} \ \forall n \ge 1,$$

and

$$\Psi(\alpha) = \sup_{s\in\mathcal{S}} \sum_{s} \phi(\cdot,\alpha), \ \forall \alpha \in (\hat{\alpha},1],$$

where

$$\sum_{s} \phi(\cdot,\alpha) \equiv \sum_{i=1}^{n} \phi(s_i,\alpha), \ \forall s = (s_i)_{1\le i\le n} \in \mathcal{S}, \ \forall \alpha \in (0,1].$$

Clearly, as $\phi(x,\alpha) \ge 0$ for all x, we have that $\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}} \sum_s \phi(\cdot,\alpha)$. Next, we claim that $\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}^4} \sum_s \phi(\cdot,\alpha)$. To prove this, we show that, for every $s \in \bar{\mathcal{S}}$, there exists $s' \in \bar{\mathcal{S}}^4$ such that

$$\sum_{s} \phi(\cdot,\alpha) \le \sum_{s'} \phi(\cdot,\alpha).$$

If s belongs to $\mathcal{S}^n$ for some $n \le 4$, or, more generally, if s has at most four components different from zero, then this is obvious. Assume instead that s has five or more components different from zero. Assume without loss of generality that $s \in \bar{\mathcal{S}}^n$ for some $n \ge 5$, that $s_i > 0$ for every i, and that the components of s_i have been sorted in increasing order. We construct s' by induction.

Let us first define a function ξ, which takes as argument a profile of market shares $\tilde{s} \in \bar{\mathcal{S}}^m$ sorted in increasing order and with strictly positive components, and returns a profile of market shares $\xi(\tilde{s})$ sorted in increasing order and with strictly positive components, such that either $\xi(\tilde{s}) \in \bar{\mathcal{S}}^m$, or $\xi(\tilde{s}) \in \bar{\mathcal{S}}^{m-1}$. $\xi(\tilde{s})$ is defined as follows:

- If $\tilde{s}_2 \ge \hat{s}(\alpha)$ (or if $\tilde{s} \in \mathcal{S}^1$), then $\xi(\tilde{s}) = \tilde{s}$.

- If $\tilde{s}_2 < \hat{s}(\alpha)$, then do the following:

– If $\tilde{s}_1 + \tilde{s}_2 \leq \hat{s}(\alpha)$, then form the $(m-1)$-dimensional vector with first component $\tilde{s}_1 + \tilde{s}_2$ and remaining components $(\tilde{s}_i)_{3\leq i\leq m}$, and sort that vector in increasing order to obtain $\xi(\tilde{s})$.

– If instead $\tilde{s}_1 + \tilde{s}_2 > \hat{s}(\alpha)$, then form the m-dimensional with first component $\tilde{s}_1 + \tilde{s}_2 - \hat{s}(\alpha)$, second component $\hat{s}(\alpha)$, and remaining components $(\tilde{s}_i)_{3\leq i\leq m}$, and sort that vector in increasing order to obtain $\xi(\tilde{s})$.

Note that, as $\phi_\alpha(\cdot)$ is convex on $[0, \hat{s}(\alpha)]$, we have that, for every $\tilde{s} \in \bar{\mathcal{S}}$

$$\sum_{\tilde{s}} \phi(\cdot, \alpha) \leq \sum_{\xi(\tilde{s})} \phi(\cdot, \alpha).$$

We can now define the sequence $(s^k)_{k\geq 0}$ by induction: $s^0 = s$; $s^{k+1} = \xi(s^k)$ for every $k \geq 0$. Let m^k denote the number of components of s^k greater or equal to $\hat{s}(\alpha)$, and n^k denote the dimensionality of the vector s^k. By definition of ξ and of the sequence $(s^k)_{k\geq 0}$, the sequence of integers $(m^k)_{k\geq 0}$ (resp. $(n^k)_{k\geq 0}$) is non-decreasing (resp. non-increasing) and bounded above by n (resp. bounded below by 1). Therefore, those sequences of integers are eventually stationary: there exists $K \geq 0$ such that $m^k = m^{k+1}$ and $n^k = n^{k+1}$ for every $k \geq K$. It follows that $(s^k)_{k\geq 0}$ is also stationary after K. Let s' be the stationary value of the sequence $(s^k)_{k\geq 0}$. Then, by induction on k,

$$\sum_{s} \phi(\cdot, \alpha) \leq \sum_{s'} \phi_\alpha(\cdot, \alpha).$$

Moreover, s' has at most one component in $[0, \hat{s}(\alpha))$ (for otherwise, $\xi(s')$ would not be equal to s'). Let n' be the dimensionality of the vector s'. We claim that $n' \leq 4$. Suppose $n' > 1$. Then,

$$1 = \sum_{i=1}^{n'} s'_i \geq (n'-1)\hat{s}(\alpha) > \frac{1}{4} \times (n'-1),$$

where the last inequality follows by Lemma 8. Hence, $n' \leq 4$. Having constructed s', we can conclude that

$$\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}^4} \sum_{s} \phi_\alpha(\cdot, \alpha). \tag{29}$$

By continuity of $\phi(\cdot, \alpha)$ (or, rather, of $\phi(\cdot, \alpha)$'s continuous extension to $[0, 1]$) and compactness of $\bar{\mathcal{S}}^4$, the maximization problem defined in equation (29) has a solution. Let s be such a solution. Then, by the convexity argument used in the construction of s', s has a most one component in $(0, \hat{s}(\alpha))$. Moreover, as $\phi(\cdot, \alpha)$ is strictly concave on $[\hat{s}(\alpha), 1]$, the components of s that are greater or equal to $\hat{s}(\alpha)$ must be equal to each other. It follows that

$$\Psi(\alpha) = \max_{x\in[0,1]} \max\left(\phi(x, \alpha) + \phi(1-x, \alpha), \phi(x, \alpha) + 2\phi\left(\frac{1-x}{2}, \alpha\right), \phi(x, \alpha) + 3\phi\left(\frac{1-x}{3}, \alpha\right)\right).$$

We (analytically) solve the above maximization problem using Mathematica. We obtain:

$$\Psi(\alpha) = \begin{cases} \frac{18\alpha}{18-3\alpha-\alpha^2} & \text{if } \alpha \leq \frac{6}{7}, \\ \frac{4\alpha}{4-\alpha^2} & \text{otherwise.} \end{cases}$$

It is straightforward to check that Ψ is strictly increasing, and that $\Psi(\hat{\alpha}) < 1 < \Psi(1)$. The unique solution of equation $\Psi(\alpha) = 1$ on the interval $(\hat{\alpha}, 1]$ is $\bar{\alpha} = \frac{3}{2}(\sqrt{57} - 7)$.

We can conclude. Assume first that $\alpha \in (\hat{\alpha}, \bar{\alpha}]$. Then, for every profile of outsiders' market shares $(s^f)_{f\in\mathcal{O}}$,

$$\sum_{f\in\mathcal{O}} \phi(s^f, \alpha) < \phi\left(1 - \sum_{f\in\mathcal{O}} s^f, \alpha\right) + \sum_{f\in\mathcal{O}} \phi(s^f, \alpha) \leq \Psi(\alpha) \leq \Psi(\bar{\alpha}) = 1.$$

Therefore, any CS-decreasing merger must have a negative external effect.

Assume instead that $\alpha > \bar{\alpha}$. We first show that there exists an infinitesimal CS-decreasing merger that has a negative external effect. Let $\mathcal{O} = \{1\}$ and $\mathcal{I} = \{2, 3\}$. As $\phi(\cdot, \alpha)$ is continuous and $\phi(0, \alpha) = 0$, there exists $s \in (0, 1)$ such that $\phi(s, \alpha) < 1$. Let $T^1 = S^{-1}(s)$, $T^2 = T^3 = S^{-1}((1 - s)/2)$, and $H^0 = 0$. Then, by construction, the pre-merger equilibrium aggregator level is $H = 1$, and market shares are as follows: $s^1 = s$, $s^2 = s^3 = (1 - s)/2$. The external effect of an infinitesimal CS-decreasing merger between firms 2 and 3 is given by $\phi(s, \alpha) - 1$, which is strictly negative by construction.

Next, we claim that there exists an infinitesimal CS-decreasing merger that has a positive external effect. As $\Psi(\alpha) > 1$, there exists $(s_i)_{1\leq i\leq n} \in (0, 1]^n$ such that $\sum_{i=1}^n s_i \leq 1$ and $\sum_{i=1}^n \phi(s_i, \alpha) > 1$. By continuity, for $\varepsilon > 0$ small enough, $\sum_{i=1}^n \phi(s_i - \varepsilon, \alpha) > 1$. Let $\mathcal{O} = \{1, \ldots, n\}$, $\mathcal{I} = \{n + 1, n + 2\}$, $s'^i = s_i - \varepsilon$ for every $i \in \mathcal{O}$, $s'^i = \frac{1}{2}\left(1 - \sum_{j=1}^n s'^j\right)$ for $i \in \mathcal{I}$, $T^i = S^{-1}(s'^i)$ for every $i \in \mathcal{I} \cup \mathcal{O}$, and $H^0 = 0$. Then, by construction, an infinitesimal CS-decreasing merger between the insiders has a positive external effect.

As any CS-decreasing merger can be decomposed into the integral of infinitesimal CS-decreasing mergers, and as a CS-decreasing merger can be made infinitesimal by tweaking the post-merger type of the merged entity, the above existence results extend immediately to non-infinitesimal mergers: if $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have a positive external effect, and CS-decreasing mergers that have a negative external effect.

We now turn to the second part of the proposition.

(i) It is easy to show that $s^* \equiv \inf_{\alpha\in[\bar{\alpha},1]} s^*(\alpha) \simeq 0.68$, where $s^*(\alpha)$ was defined in Lemma 8. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ such that $s \geq_1 s'$, and $s^f \leq s^*$ for every $f \in \mathcal{O}$. There exists an injection $\kappa : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $s^{\kappa(f)} \geq s'^f$ for every $f \in \mathcal{O}'$. Note that

$$-1 + \sum_{f\in\mathcal{O}'} \phi(s'^f) \leq -1 + \sum_{f\in\mathcal{O}'} \phi(s^{\kappa(f)}) \leq -1 + \sum_{f\in\mathcal{F}} \phi(s^f, \alpha),$$

where the first inequality follows by Lemma 8, and the second inequality follows by injectivity of κ and non-negativity of ϕ.

(ii) It is easy to show that $\hat{s} \equiv \inf_{\alpha \in [\bar{\alpha},1]} \hat{s}(\alpha) \simeq 0.29$, where $\hat{s}(\alpha)$ was defined in Lemma 8. Let $s = (s^f)_{f \in \mathcal{O}}$ and $s' = (s'^f)_{f \in \mathcal{O}'}$ such that $s \geq_2 s'$, $s^f \leq \hat{s}$ for every $f \in \mathcal{O}$, and $s'^f \leq \hat{s}$ for every $f \in \mathcal{O}'$. As $s \geq_2 s'$, those vectors have the same length, and we can assume that $\mathcal{O} = \mathcal{O}' = \{1, \ldots, n\}$ without loss of generality. Note that

$$\begin{aligned} -1 + \sum_{f=1}^{n} \phi(s^f, \alpha) &= -1 + n \int_0^{\hat{s}} \phi(x, \alpha) dP_s(x), \\ &\geq -1 + n \int_0^{\hat{s}} \phi(x, \alpha) dP_{s'}(x), \\ &= -1 + \sum_{f=1}^{n} \phi(s'^f, \alpha), \end{aligned}$$

where the inequality follows from the convexity of $\phi(\cdot, \alpha)$ on $[0, \hat{s}]$ (see Lemma 8), and the fact that $\int_0^{\hat{s}} x dP_s(x) = \int_0^{\hat{s}} x dP_{s'}(x)$ and $P_{s'}$ second-order stochastically dominates P_s. $\square$

F Proof of Proposition 8

We prove a series of lemmas that jointly imply Proposition 8.

We begin by approximating pre-merger consumer surplus:

Lemma 9. *The pre-merger level of the aggregator is* $H^*(s) = \frac{H^0}{1 - \sum_{g \in \mathcal{F}} s^g}$. *Moreover, in the neighborhood of* $s = 0$,

$$CS(s) = \log H^0 + \sum_{f \in \mathcal{F}} s^f + \frac{1}{2} \left(\sum_{f \in \mathcal{F}} s^f \right)^2 + o\left(\|s\|^2\right).$$

Proof. The first part of the lemma follows immediately from the equilibrium condition

$$\frac{H^0}{H^*} + \sum_{g \in \mathcal{F}} s^g = 1.$$

The second part of the lemma follows from the fact that, in the neighborhood of $x = 0$,

$$-\log(1 - x) = x + \frac{1}{2} x^2 + o(x^2). \qquad \square$$

Next, we compute the first and second (cross-)partial derivatives of the type vector $T(s)$:

Lemma 10. *For every* $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{otherwise.} \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\left.\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} 2(1+\alpha) & \text{if } f = f' = f'', \\ 0 & \text{if } f' \neq f \text{ and } f'' \neq f, \\ \frac{H^0}{S'(0)} & \text{otherwise.} \end{cases}$$

Proof. Let $f \in \mathcal{F}$. As $s^f = S\left(\frac{T^f}{H^*(s)}\right)$, we have that

$$T^f = H^* S^{-1}(s^f) = H^0 \frac{\Theta(s^f)}{1 - \sum_{g \in \mathcal{F}} s^g} \equiv H^0 \Theta(s^f) \Psi(s),$$

where we have used the inverse function Θ that was defined in Lemma 6.

Note that, for every $(f, f', f'') \in \mathcal{F}^3$,

$$\Psi(0) = \left.\frac{\partial \Psi}{\partial s^f}\right|_{s=0} = 1,$$
$$\left.\frac{\partial^2 \Psi}{\partial s^f \partial s^{f'}}\right|_{s=0} = 2.$$

Therefore, for every $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = H^0 \left.\left(\frac{\partial \Theta(s^f)}{\partial s^{f'}} \Psi(s) + \Theta(s^f) \frac{\partial \Psi}{\partial s^{f'}} \right)\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{if } f \neq f'. \end{cases}$$

Finally, for every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned} \left.\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= H^0 \left.\left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}} \Psi(s) + \Theta(s^f) \frac{\partial^2 \Psi}{\partial s^{f'} \partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}} \frac{\partial \Psi}{\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}} \frac{\partial \Psi}{\partial s^{f'}} \right)\right|_{s=0}, \\ &= H^0 \left.\left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}} \right)\right|_{s=0}, \\ &= H^0 \times \begin{cases} \Theta''(0) + 2\Theta'(0) & \text{if } f = f' = f'', \\ 0 & \text{if } f', f'' \neq f, \\ \Theta'(0) & \text{otherwise,} \end{cases} \end{aligned}$$

$$= \frac{H^0}{S'(0)} \times \begin{cases} 2(\alpha+1) & \text{if } f = f' = f'', \\ 0 & \text{if } f', f'' \neq f, \\ 1 & \text{otherwise.} \end{cases}$$

□

We now use Lemma 10 to obtain a second-order Taylor approximation of $T^f(s)$ in the neighborhood of $s = 0$:

Lemma 11. *In the neighborhood of $s = 0$,*

$$T^f(s) = \frac{H^0}{S'(0)} \left(s^f + \alpha (s^f)^2 + s^f \sum_{g \in \mathcal{F}} s^g \right) + o(\|s\|^2).$$

Proof. By Lemma 10, first-order terms are simply given by $\frac{H^0}{S'(0)} s^f$. Second-order terms are given by

$$\frac{H^0}{S'(0)} \frac{1}{2} \left(2(1+\alpha)(s^f)^2 + 2 s^f \sum_{g \neq f} s^g \right) = \frac{H^0}{S'(0)} \left(\alpha (s^f)^2 + s^f \sum_{g \in \mathcal{F}} s^g \right).$$

The lemma follows by Taylor's theorem. □

To ease notation, let $\overline{H}(s) \equiv H^*(\bar{s}(s))$ be the post-merger level of the aggregator. We now provide an approximation of the market power effect of the merger, measured in terms of consumer surplus—the first part of Proposition 8:

Lemma 12. *In the neighborhood of $s = 0$,*

$$CS(\bar{s}(s)) - CS(s) = -\alpha \, \Delta HHI(s) + o(\|s\|^2).$$

Proof. By definition of $\overline{H}$, we have that

$$\frac{H^0}{\overline{H}} + \sum_{g \in \overline{\mathcal{F}}} S\left(\frac{T^g}{\overline{H}} \right) = 1.$$

Totally differentiating this expression, we obtain:

$$-\frac{d\overline{H}}{\overline{H}} \left(\frac{H^0}{\overline{H}} + \sum_{g \in \overline{\mathcal{F}}} \frac{T^g}{\overline{H}} S'\left(\frac{T^g}{\overline{H}} \right) \right) + \frac{1}{\overline{H}} \sum_{g \in \overline{\mathcal{F}}} S'\left(\frac{T^g}{\overline{H}} \right) \sum_{f \in \mathcal{F}} \frac{\partial T^g}{\partial s^f} ds^f = 0.$$

Hence,

$$\frac{\partial \overline{H}}{\partial s^f} = \overline{H} \frac{\sum_{g \in \overline{\mathcal{F}}} S'\left(\frac{T^g}{\overline{H}} \right) \frac{\partial T^g}{\partial s^f}}{H^0 + \sum_{g \in \overline{\mathcal{F}}} T^g S'\left(\frac{T^g}{\overline{H}} \right)}.$$

Hence, by Lemma 11 and as $T^M = \sum_{g\in\mathcal{M}} T^g$,

$$\left.\frac{\partial \overline{H}}{\partial s^f}\right|_{s=0} = H^0.$$

Next, we compute the Hessian of $\overline{H}$. Note that, for every $f, f' \in \mathcal{F}$

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \frac{\partial \overline{H}}{\partial s^{f'}} \times 1 + H^0 \times \frac{1}{(H^0)^2}\left(\left(\sum_{g\in\overline{\mathcal{F}}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right)\right) H^0 \right.\\
&\qquad \left. -H^0\left(\sum_{g\in\overline{\mathcal{F}}}\frac{\partial T^g}{\partial s^f} S'(0)\right)\right),\\
&= H^0 + \sum_{g\in\overline{\mathcal{F}}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right) - \sum_{g\in\overline{\mathcal{F}}}\frac{\partial T^g}{\partial s^f} S'(0),\\
&= \sum_{g\in\overline{\mathcal{F}}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right),\\
&= \left(\frac{\partial^2 T^M}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^M}{\partial s^f}\frac{\partial T^M}{\partial s^{f'}} S''(0)\right)\\
&\qquad + \sum_{g\in\mathcal{O}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right).
\end{aligned}$$

Assume first that $f \in \mathcal{O}$ and/or $f' \in \mathcal{O}$. Then, by Lemma 11 and as $T^M = \sum_{g\in\mathcal{M}} T^g$,

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \begin{cases} 2H^0 & \text{if } f \neq f', \\ \frac{H^0}{S'(0)} 2(1+\alpha) S'(0) + \frac{1}{H^0}\left(\frac{H^0}{S'(0)}\right)^2 S''(0) & \text{if } f = f', \end{cases}\\
&= 2H^0.
\end{aligned}$$

Next, assume instead that $f, f' \in \mathcal{M}$. Then,

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \begin{cases} 2H^0 + \frac{1}{H^0}\left(\frac{H^0}{S'(0)}\right)^2 S''(0) & \text{if } f \neq f', \\ \frac{H^0}{S'(0)} 2(1+\alpha) S'(0) + \frac{1}{H^0}\left(\frac{H^0}{S'(0)}\right)^2 S''(0) & \text{if } f = f', \end{cases}\\
&= \begin{cases} 2H^0(1-\alpha) & \text{if } f \neq f', \\ 2H^0 & \text{if } f = f'. \end{cases}
\end{aligned}$$

By Taylor's theorem,

$$\begin{aligned}\overline{H}(s) &= H^0 + H^0 \sum_{f\in\mathcal{F}} s^f + \frac{H^0}{2}\left(2\sum_{f,g\in\mathcal{F}} s^f s^g - 2\alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g\right) + o(\|s\|^2),\\ &= H^0\left(1 + \sum_{f\in\mathcal{F}} s^f + \left(\sum_{f\in\mathcal{F}} s^f\right)^2 - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g\right) + o(\|s\|^2).\end{aligned}$$

Using the fact that $\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$ in the neighborhood of $x=0$, this implies that

$$\begin{aligned}\log \overline{H}(s) &= \log H^0 + \sum_{f\in\mathcal{F}} s^f + \left(\sum_{f\in\mathcal{F}} s^f\right)^2 - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g - \frac{1}{2}\left(\sum_{f\in\mathcal{F}} s^f\right)^2 + o(\|s\|^2),\\ &= \log H^*(s) - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g + o(\|s\|^2), \text{ by Lemma 9},\\ &= \log H^*(s) - \alpha\, \Delta\mathrm{HHI}(s) + o(\|s\|^2).\end{aligned}$$

□

Next, we approximate post-merger market shares:

Lemma 13. *In the neighborhood of $s=0$, for every $f \in \mathcal{O}$*

$$\bar{s}^f = s^f + o(\|s\|^2),$$

and

$$\bar{s}^M = \sum_{f\in\mathcal{M}} s^f - \alpha\, \Delta HHI(s) + o(\|s\|^2).$$

Proof. By definition, for every $f \in \overline{\mathcal{F}}$, $\bar{s}^f = S\left(T^f/\overline{H}\right)$. For every $f' \in \mathcal{F}$, we have:

$$\frac{\partial \bar{s}^f}{\partial s^{f'}} = \frac{1}{\overline{H}}\left(\frac{\partial T^f}{\partial s^{f'}} - \frac{T^f}{\overline{H}}\frac{\partial \overline{H}}{\partial s^{f'}}\right) S'\left(\frac{T^f}{\overline{H}}\right).$$

It follows that

$$\left.\frac{\partial \bar{s}^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} 0 & \text{if } f\neq f' \text{ and } (f\neq M \text{ or } f'\notin\mathcal{M}),\\ 1 & \text{otherwise.}\end{cases}$$

For every $f \in \overline{\mathcal{F}}$ and $f', f'' \in \mathcal{F}$,

$$\left.\frac{\partial^2 \bar{s}^f}{\partial s^{f'}\partial s^{f''}}\right|_{s=0} = -\frac{\partial \overline{H}}{\partial s^{f''}}\frac{1}{\overline{H}^2}\frac{\partial T^f}{\partial s^{f'}} S'(0) + \frac{1}{\overline{H}}\left(\frac{\partial^2 T^f}{\partial s^{f'}\partial s^{f''}} - \frac{1}{\overline{H}}\frac{\partial T^f}{\partial s^{f''}}\frac{\partial \overline{H}}{\partial s^{f'}}\right) S'(0)$$

$$
\begin{aligned}
&+\frac{1}{\overline{H}^2}\frac{\partial T^f}{\partial s^{f'}}\frac{\partial T^f}{\partial s^{f''}}S''(0),\\
&= -\frac{1}{H^0}\frac{\partial T^f}{\partial s^{f'}}S'(0)+\frac{1}{H^0}\left(\frac{\partial^2 T^f}{\partial s^{f'}\partial s^{f''}}-\frac{\partial T^f}{\partial s^{f''}}\right)S'(0)+\frac{1}{(H^0)^2}\frac{\partial T^f}{\partial s^{f'}}\frac{\partial T^f}{\partial s^{f''}}S''(0),\\
&= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^f}{\partial s^{f'}\partial s^{f''}}-\frac{\partial T^f}{\partial s^{f'}}-\frac{\partial T^f}{\partial s^{f''}}\right)+\frac{S''(0)}{(H^0)^2}\frac{\partial T^f}{\partial s^{f'}}\frac{\partial T^f}{\partial s^{f''}}.
\end{aligned}
$$

Suppose first that $f \neq M$, so that $f \in \mathcal{F}$. Clearly, if $f' \neq f$ and $f'' \neq f$, then,

$$
\left.\frac{\partial^2 \overline{s}^f}{\partial s^{f'}\partial s^{f''}}\right|_{s=0} = 0.
$$

If $f'' \neq f$, then

$$
\left.\frac{\partial^2 \overline{s}^f}{\partial s^{f}\partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^f}{\partial s^{f}\partial s^{f''}}-\frac{\partial T^f}{\partial s^f}\right) = 0.
$$

Finally,

$$
\begin{aligned}
\left.\frac{\partial^2 \overline{s}^f}{\partial (s^{f})^2}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^f}{\partial (s^{f})^2}-2\frac{\partial T^f}{\partial s^f}\right)+\frac{S''(0)}{(H^0)^2}\left(\frac{\partial T^f}{\partial s^f}\right)^2,\\
&= \frac{S'(0)}{H^0}\left(\frac{H^0}{S'(0)}2(1+\alpha)-2\frac{H^0}{S'(0)}\right)+\frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2,\\
&= 0.
\end{aligned}
$$

Next, assume that $f = M$. Clearly, if $f', f'' \notin \mathcal{M}$, then

$$
\left.\frac{\partial^2 \overline{s}^M}{\partial s^{f'}\partial s^{f''}}\right|_{s=0} = 0.
$$

Next assume that $f'' \notin \mathcal{M}$ and $f' \in \mathcal{M}$. Then,

$$
\begin{aligned}
\left.\frac{\partial^2 \overline{s}^M}{\partial s^{f'}\partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'}\partial s^{f''}}-\frac{\partial T^M}{\partial s^{f'}}\right),\\
&= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial s^{f'}\partial s^{f''}}-\frac{\partial T^{f'}}{\partial s^{f'}}\right),\\
&= 0.
\end{aligned}
$$

Next, assume that $f', f'' \in \mathcal{M}$. Then,

$$
\begin{aligned}
\left.\frac{\partial^2 \overline{s}^M}{\partial s^{f'}\partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'}\partial s^{f''}}-\frac{\partial T^{f'}}{\partial s^{f'}}-\frac{\partial T^{f''}}{\partial s^{f''}}\right)+\frac{S''(0)}{(H^0)^2}\frac{\partial T^{f'}}{\partial s^{f'}}\frac{\partial T^{f''}}{\partial s^{f''}},\\
&= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'}\partial s^{f''}}-2\frac{H^0}{S'(0)}\right)+\frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2.
\end{aligned}
$$

Hence, if $f' = f''$, then

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial (s^{f'})^2} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2, \\ = 0.$$

If instead $f' \neq f''$, then

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial s^{f'} \partial s^{f''}} + \frac{\partial^2 T^{f''}}{\partial s^{f'} \partial s^{f''}} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2, \\ = -2\alpha.$$

The lemma follows by Taylor's theorem. □

Let

$$\Pi(s) = \sum_{f \in \mathcal{F}} \left(\frac{1}{1-\alpha s^f} - 1\right) \text{ and } \overline{\Pi}(s) = \sum_{f \in \overline{\mathcal{F}}} \left(\frac{1}{1-\alpha \bar{s}^f} - 1\right)$$

be aggregate profits, pre- and post-merger, respectively.

Lemma 14. *In the neighborhood of $s = 0$,*

$$\overline{\Pi}(s) - \Pi(s) = o(\|s\|^2).$$

Proof. By Lemma 13, and as $\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + o(\|x\|^2)$ in the neighborhood of $x = 0$, we have that

$$\Pi(s) = \alpha \sum_{f \in \mathcal{F}} s^f + \alpha^2 \sum_{f \in \mathcal{F}} (s^f)^2 + o(\|s\|^2),$$

and

$$\begin{aligned}
\overline{\Pi}(s) &= \frac{1}{1-\alpha \bar{s}^M} - 1 + \sum_{f \in \mathcal{O}} \left(\frac{1}{1-\alpha \bar{s}^f} - 1\right), \\
&= \alpha \left(\sum_{f \in \mathcal{M}} s^f - \alpha \sum_{\substack{f,g \in \mathcal{M} \\ f \neq g}} s^f s^g\right) + \alpha^2 \left(\sum_{f \in \mathcal{M}} s^f\right)^2 + \alpha \sum_{f \in \mathcal{O}} s^f + \alpha^2 \sum_{f \in \mathcal{O}} (s^f)^2 + o(\|s\|^2), \\
&= \alpha \sum_{f \in \mathcal{F}} s^f + \alpha^2 \sum_{f \in \mathcal{F}} (s^f)^2 + o(\|s\|^2), \\
&= \Pi(s) + o(\|s\|^2).
\end{aligned}$$

□

Combining Lemmas 12 and 14 proves the second part of Proposition 8:

Lemma 15. *In the neighborhood of $s = 0$,*

$$AS(\bar{s}(s)) - AS(s) = -\alpha\, \Delta HHI(s) + o(\|s\|^2).$$

G Approximation Results Around Monopolistic Competition Conduct

This appendix is organized as follows. We first provide more details on our treatment of firm conduct in Appendix G.1. We then prove Proposition 9 in Appendix G.2.

G.1 Firm Conduct

In this subsection, we derive fitting-in functions for any conduct parameter θ in the more general framework with broad firms, as introduced in Appendix B. (The case without nests, studied in the main text, can simply be obtained by setting $\beta = 1$ in what follows.) The first-order condition for product $i \in n \in f$ is given by

$$\frac{H_n^{\beta-1}}{H}\left(-h_i' - (p_i - c_i)h_i'' + (1-\beta)\frac{\partial H_n}{\partial p_i}\frac{\sum_{j\in n}(p_j - c_j)h_j'}{H_n}\right.$$
$$\left. + \theta \times \frac{H_n^{1-\beta}}{H}\frac{\partial H}{\partial p_i}\sum_{l\in f} H_l^{\beta-1}\sum_{j\in l}(p_j - c_j)h_j'\right) = 0,$$

which can be rewritten as

$$\frac{p_i - c_i}{p_i}\frac{p_i h_i''}{-h_i'} = 1 + (1-\beta)\frac{\sum_{j\in n}(p_j - c_j)(-h_j')}{H_n} + \theta\beta\frac{1}{H}\sum_{l\in f} H_l^{\beta-1}\sum_{j\in l}(p_j - c_j)(-h_j'), \quad (30)$$

so that the common ι-markup property within nest n continues to hold. Let $\tilde{\mu}_n$ be firm f's ι-markup in nest n. Then, using equation (18), equation (30) simplifies to

$$\tilde{\mu}_n\left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \theta\tilde{\alpha}\beta\frac{1}{H}\sum_{l\in f}\tilde{\mu}^l H_l^{\beta}, \quad (31)$$

so that $\tilde{\mu}_n = \tilde{\mu}_{n'} \equiv \tilde{\mu}^f$ for every $n, n' \in f$. Using the common ι-markup property both within nest and across nests allows us to further simplify equation (31):

$$\tilde{\mu}^f\left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \theta\tilde{\alpha}\beta\tilde{\mu}^f s^f.$$

Defining $\mu^f \equiv \tilde{\mu}^f\left(1 - \tilde{\alpha}(1-\beta)\right)$ as we did in Appendix B, this implies that

$$\mu^f = \frac{1}{1 - \theta\alpha s^f}. \quad (32)$$

As the conduct parameter θ does not affect the demand system, it is still the case that

$$s^f = \begin{cases} \frac{T^f}{H}\left(1-(1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{under NCES demand,} \\ \frac{T^f}{H}\,\mathrm{e}^{-\mu^f} & \text{under NMNL demand.} \end{cases} \tag{33}$$

Thus, firm f's markup and market share jointly solve equations (32) and (33). This pins down the fitting-in functions $m(T^f/H,\theta)$ and $S(T^f/H,\theta)$. The profit fitting-in function is given by

$$\begin{aligned}
\pi(T^f/H,\theta) &= \frac{\beta}{H}\sum_{l\in f} H_l^{\beta-1}\sum_{j\in f}(p_j-c_j)(-h_j'), \\
&= \frac{\beta}{H}\tilde{\mu}^f\tilde{\alpha}\sum_{l\in f} H_l^{\beta}, \text{ using equation (18),} \\
&= \alpha\mu^f s^f, \text{ by definition of } \mu^f,\ s^f, \text{ and } \alpha, \\
&= \alpha m\left(\frac{T^f}{H},\theta\right) S\left(\frac{T^f}{H},\theta\right), \\
&= \frac{\alpha S\left(\frac{T^f}{H},\theta\right)}{1-\alpha\theta S\left(\frac{T^f}{H},\theta\right)}.
\end{aligned}$$

The equilibrium aggregator level $H^*(\theta)$ uniquely solves the equation

$$\frac{H^0}{H} + \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H},\theta\right) = 1. \tag{34}$$

It is easy to see that $H^*(\theta)$, $m(\cdot,\theta)$, $S(\cdot,\theta)$, and $\pi(\cdot,\theta)$ all tend to their value under monopolistic competition as θ tends to 0, and to their value under fully-fledged oligopoly as θ tends to 1.

G.2 Proof of Proposition 9

We begin by proving a series of preliminary lemmas. Computing the partial derivatives of $S(x,\theta)$ at $\theta = 0$, we obtain:

Lemma 16. *For every $\alpha \in (0,1]$ and $x > 0$,*

$$\begin{aligned}
\left.\frac{\partial S}{\partial x}\right|_{(x,0)} &= \frac{S(x,0)}{x}, \\
\textit{and}\ \left.\frac{\partial S}{\partial \theta}\right|_{(x,0)} &= -\alpha S(x,0)^2.
\end{aligned}$$

Proof. Under NMNL demand,

$$S = x\,\mathrm{e}^{-m} = x \exp\left(-\frac{1}{1-\theta S}\right).$$

Hence, at $\theta = 0$,

$$dS = \frac{S}{x}dx - S^2 d\theta,$$

which proves the lemma for the case $\alpha = 1$.

Under NCES demand,

$$S = x\,(1-(1-\alpha)m)^{\frac{\alpha}{1-\alpha}} = x\left(1 - \frac{1-\alpha}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}} = x\left(\alpha\frac{1-\theta S}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}}.$$

Hence, at $\theta = 0$,

$$\begin{aligned} dS &= \frac{S}{x}dx + \frac{\alpha}{1-\alpha}S\frac{1-\alpha\theta S}{1-\theta S}\frac{1}{(1-\alpha\theta S)^2}\Big(\left(-\theta(1-\alpha\theta S)+\alpha\theta(1-\theta S)\right)dS \\ &\quad + \left(-S(1-\alpha\theta S)+\alpha S(1-\theta S)\right)d\theta\Big), \\ &= \frac{S}{x}dx - \alpha S^2, \end{aligned}$$

which proves the lemma for the case $\alpha < 1$. □

Next, we relate pre-merger consumer surplus CS(θ) to the pre-merger Herfindahl index HHI(θ):

Lemma 17. *The following holds:*

$$\left.\frac{d\log H^*}{d\theta}\right|_{\theta=0} = -\alpha\, HHI(0).$$

This implies that, in the neighborhood of $\theta = 0$,

$$CS(\theta) - CS(0) = -\alpha\, HHI(\theta)\theta + o(\theta).$$

Proof. Totally differentiating equilibrium condition (34), we obtain:

$$-\frac{dH^*}{H^*}\left(\frac{H^0}{H^*} + \sum_{f\in\mathcal{F}}\frac{T^f}{H^*}\frac{\partial S}{\partial(T^f/H^*)}\left(\frac{T^f}{H^*},\theta\right)\right) + d\theta\sum_{f\in\mathcal{F}}\frac{\partial S}{\partial\theta}\left(\frac{T^f}{H^*},\theta\right) = 0.$$

Evaluating the above expression at $\theta = 0$, and using Lemma 16 and the equilibrium condition, we obtain:

$$-\frac{dH^*}{H^*(0)} - d\theta\sum_{f\in\mathcal{F}}\alpha S\left(\frac{T^f}{H^*(0)},0\right)^2 = 0,$$

which proves the first part of the lemma.

The second part of the lemma follows by Taylor's theorem:

$$\mathrm{CS}(\theta) - \mathrm{CS}(0) = -\alpha\,\mathrm{HHI}(0)\theta + o(\theta) = -\alpha\,\mathrm{HHI}(\theta)\theta + o(\theta),$$

where the second equality follows as $\mathrm{HHI}(\theta) - \mathrm{HHI}(0)$ is at most first order. $\square$

Let $\Pi(\theta)$ denote aggregate equilibrium profits when the conduct parameter is θ. We compute $\Pi'(0)$:

Lemma 18. $\Pi'(0) = \alpha^2\, \mathit{HHI}(0) \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(0)}, 0\right)$.

Proof. Let $\pi^f(\theta) = \alpha s^f(\theta)/(1-\alpha\theta s^f(\theta))$ denote firm f's equilibrium profit. Note that

$$\begin{aligned} s^{f\prime}(0) &= \left(-\frac{T^f}{H^*}\frac{d\log H^*}{d\theta}\frac{\partial S}{\partial(T^f/H^*)} + \frac{\partial S}{\partial\theta}\right)\Bigg|_{\theta=0}, \\ &= \alpha\,\mathrm{HHI}(0)s^f(0) - \alpha(s^f(0))^2. \end{aligned}$$

Hence,

$$\pi^{f\prime}(0) = \alpha\left(s^{f\prime}(0) - s^f(0)\left(-\alpha s^f(0)\right)\right) = \alpha^2\,\mathrm{HHI}(0)s^f(0).$$

Adding up over all firms proves the lemma. $\square$

Combining Lemmas 17 and 18, we obtain an approximation of pre-merger aggregate surplus around monopolistic competition conduct:

Lemma 19. *In the neighborhood of $\theta = 0$,*

$$\mathit{AS}(\theta) - \mathit{AS}(0) = -\alpha\,\mathit{HHI}(\theta)\left(1 - \sum_{f\in\mathcal{F}} s^f(\theta)\right)\theta + o(\theta).$$

Proof. Lemmas 17 and 18 and Taylor's theorem imply that

$$\mathrm{AS}(\theta) - \mathrm{AS}(0) = -\alpha\,\mathrm{HHI}(0)\left(1 - \sum_{f\in\mathcal{F}} s^f(0)\right)\theta + o(\theta).$$

The lemma follows from the fact that

$$\mathrm{HHI}(0)\left(1 - \sum_{f\in\mathcal{F}} s^f(0)\right) - \mathrm{HHI}(\theta)\left(1 - \sum_{f\in\mathcal{F}} s^f(\theta)\right)$$

is at most first order. $\square$

We are now in a position to prove the proposition:

Proof of Proposition 9. In the following, $\overline{X}(\theta)$ refers to the post-merger level of the variable $X(\theta)$. Let

$$\Sigma(\theta) \equiv \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(\theta)}, \theta\right)$$

be the firms' aggregate pre-merger market share. Note that $\mathrm{CS}(0) = \overline{\mathrm{CS}}(0)$, $\mathrm{AS}(0) = \overline{\mathrm{AS}}(0)$, $H^*(0) = \overline{H}^*(0)$, $\Sigma(0) = \overline{\Sigma}(0)$, and $\overline{\mathrm{HHI}}(0) - \mathrm{HHI}(0) = \Delta\mathrm{HHI}(0)$.

Using these equalities and Lemma 17, we obtain:

$$\begin{aligned}
\overline{\mathrm{CS}}(\theta) - \mathrm{CS}(\theta) &= -\alpha\left(\overline{\mathrm{HHI}}(\theta) - \mathrm{HHI}(\theta)\right)\theta + o(\theta), \\
&= -\alpha\left(\overline{\mathrm{HHI}}(0) - \mathrm{HHI}(0) + o(1)\right)\theta + o(\theta), \\
&= -\alpha\,\Delta\mathrm{HHI}(0)\theta + o(\theta), \\
&= -\alpha\left(\Delta\mathrm{HHI}(\theta) + o(1)\right)\theta + o(\theta), \\
&= -\alpha\,\Delta\mathrm{HHI}(\theta)\theta + o(\theta),
\end{aligned}$$

which proves the first part of the proposition.

Similarly, using Lemma 19, we obtain:

$$\begin{aligned}
\overline{\mathrm{AS}}(\theta) - \mathrm{AS}(\theta) &= -\alpha\Big(\overline{\mathrm{HHI}}(\theta)\left(1 - \alpha\overline{\Sigma}(\theta)\right) - \mathrm{HHI}(\theta)\left(1 - \alpha\Sigma(\theta)\right)\Big)\theta + o(\theta), \\
&= -\alpha\Big(\overline{\mathrm{HHI}}(0)\left(1 - \alpha\Sigma(0)\right) - \mathrm{HHI}(0)\left(1 - \alpha\Sigma(0)\right) + o(1)\Big)\theta + o(\theta), \\
&= -\alpha\left(1 - \alpha\Sigma(0)\right)\left(\overline{\mathrm{HHI}}(0) - \mathrm{HHI}(0)\right)\theta + o(\theta), \\
&= -\alpha\left(1 - \alpha\Sigma(\theta) + o(1)\right)\left(\Delta\mathrm{HHI}(\theta) + o(1)\right)\theta + o(\theta), \\
&= -\alpha\left(1 - \alpha\Sigma(\theta)\right)\Delta\mathrm{HHI}(\theta)\theta + o(\theta),
\end{aligned}$$

which proves the second part of the proposition. $\square$

H Proof of Proposition 10

Proof. Fix a profile of prices p^{-f} for firm f's rivals, and let $\mathcal{N}^f = \bigcup_{l\in f} l$. Define

$$H^{0\prime} = H^0 + \sum_{g\in\mathcal{F}\setminus\{f\}} \sum_{l\in g} \left(\sum_{j\in l} h_j(p_j^{-f})\right)^\beta > 0,$$

and

$$G(p) = \beta\frac{\sum_{l\in f}\left(\sum_{i\in l} h_i(p_i)\right)^{\beta-1}\sum_{j\in l}(p_j - c_j)(-h_j'(p_j))}{H^{0\prime} + \sum_{l\in f}\left(\sum_{i\in l} h_i(p_i)\right)^\beta},$$

for every profile of prices $p = (p_j)_{j\in\mathcal{N}^f}$. Note that $G(p)$ is the profit firm f receives when it sets the price vector p and its rivals rivals set the price vector p^{-f}. Our goal is to show that the maximization problem

$$\max_{p \in \mathbb{R}_{++}^{\mathcal{N}^f}} G(p)$$

has a unique solution, and that the price vector p solves that maximization problem if and only if it satisfies the first-order conditions.

The proof follows a similar development to the proof of Lemmas B–H in the Appendix to Nocke and Schutz (2018). It proceeds as follows. We first show that pricing some (or all) of the products below cost is strictly suboptimal (Step 1). We then extend the domain of G to price vectors that have infinite components (Step 2). Combining Steps 1 and 2 allows us to show that the profit maximization problem has a solution (Step 3). We then show that there exists a unique price vector satisfying the first-order conditions of profit maximization (Step 4). Combining Steps 1–4, we can conclude that the profit maximization problem has a unique solution, and that first-order conditions are necessary and sufficient for optimality.

Step 1: No product is priced below cost. We first argue that firm f's products are substitutes. Let $n, n' \in f$ and $(i, i') \in n \times n'$ such that $i \neq i'$. If $n \neq n'$, then

$$\frac{\partial D_i}{\partial p_{i'}} = \beta^2 \frac{h_i' H_n^{\beta-1} h_{i'}' H_{n'}^{\beta-1}}{H^2} > 0.$$

If instead $n = n'$, then

$$\frac{\partial D_i}{\partial p_{i'}} = \frac{\beta h_i' h_{i'}'}{H} \left((1-\beta) H_n^{\beta-2} + \beta \frac{H_n^{2(\beta-1)}}{H} \right) > 0.$$

Let p be a price vector for firm f such that $p_j < c_j$ for some product $j \in \mathcal{N}^f$. Define a new price vector $\tilde{p}$ for firm f such that for every $i \in \mathcal{N}^f$, $\tilde{p}_i = \max(c_i, p_i)$. When firm f deviates from p to $\tilde{p}$, it stops making losses on those products that were originally priced below cost, and, by substitutability, it makes more profits on those products that were priced above cost. Therefore, price vector p is not optimal for firm f. When looking for a solution to firm f's profit maximization problem, we can therefore confine our attention to price vectors in $\prod_{j \in \mathcal{N}^f} [c_j, \infty)$.

Step 2: Defining G at infinite prices. Let $\hat{p} \in \prod_{j \in \mathcal{N}^f} [c_j, \infty]$. Suppose $\hat{p}$ has at least one infinite component, and let $(p^k)_{k \geq 0}$ be a sequence over $\prod_{j \in \mathcal{N}^f} [c_j, \infty)$ such that $p^k \underset{k \to \infty}{\longrightarrow} \hat{p}$. Let

$$f' = \{ l \in f : \exists i \in l \text{ s.t. } \hat{p}_i < \infty \}$$

and

$$\mathcal{N}^{f\prime} = \{ j \in \mathcal{N}^f : \hat{p}_j < \infty \}.$$

Clearly, as k tends to infinity, the denominator of $G(p^k)$ tends to[2]

$$H^{0\prime} + \sum_{l \in f'} \left(\sum_{j \in l \cap \mathcal{N}^{f'}} h_j(\hat{p}_j) \right)^{\beta}.$$

Next, let $i \in \mathcal{N}^f \setminus \mathcal{N}^{f'}$. Let $l \in f$ be the nest that contains product i. Note that, for every $k \geq 0$,

$$(p_i^k - c_i)(-h_i'(p_i^k)) \left(\sum_{j \in l} h_j(p_j^k) \right)^{\beta - 1} \leq (p_i^k - c_i)(-h_i'(p_i^k)) \left(h_i(p_i^k) \right)^{\beta - 1}.$$

Under NCES demand,

$$(p_i^k - c_i)(-h_i'(p_i^k)) \left(h_i(p_i^k) \right)^{\beta - 1} \leq (\sigma - 1) a_i (p_i^k)^{\beta(1-\sigma)} \underset{k \to \infty}{\longrightarrow} 0.$$

Under NMNL demand,

$$(p_i^k - c_i)(-h_i'(p_i^k)) \left(h_i(p_i^k) \right)^{\beta - 1} \leq \frac{1}{\lambda} p_i^k \exp\left(\frac{\beta}{\lambda} \left(a_i - p_i^k \right) \right) \underset{k \to \infty}{\longrightarrow} 0.$$

It follows that

$$G(p^k) \underset{k \to \infty}{\longrightarrow} \beta \frac{\sum_{l \in f'} \left(\sum_{i \in l \cap \mathcal{N}^{f'}} h_i(\hat{p}_i) \right)^{\beta - 1} \sum_{j \in l \cap \mathcal{N}^{f'}} (\hat{p}_j - c_j)(-h_j'(\hat{p}_j))}{H^{0\prime} + \sum_{l \in f'} \left(\sum_{i \in l \cap \mathcal{N}^{f'}} h_i(\hat{p}_i) \right)^{\beta}} \equiv G(\hat{p}).$$

We have thus extended the domain of G to $\prod_{j \in \mathcal{N}^f} [c_j, \infty]$. Note that, at $\hat{p}$, G has smooth partial derivatives with respect to $(p_i)_{i \in \mathcal{N}^{f'}}$.

Step 3: The profit maximization problem has a solution. By continuity of G (as established in the previous step) and compactness of $\prod_{j \in \mathcal{N}^f} [c_j, \infty]$, the maximization problem

$$\max_{p \in \prod_{j \in \mathcal{N}^f} [c_j, \infty]} G(p)$$

has a solution $\hat{p}$. Clearly, $\hat{p}$ has at least one finite component, for otherwise $G(\hat{p})$ would be equal to zero, as shown above.

Assume for a contradiction that $\hat{p}$ has some infinite components, and define f' and $\mathcal{N}^{f'}$ as in the previous step. As $\hat{p}$ maximizes G, it must be the case that $\left. \frac{\partial G}{\partial p_i} \right|_{\hat{p}} = 0$ for every $i \in \mathcal{N}^{f'}$. Manipulating the first-order conditions as we did in Appendix B, we obtain the existence of

[2] By convention, the sum of an empty collection of reals is zero.

a $\tilde{\mu}^f$ such that, for every $i \in \mathcal{N}^{f\prime}$,

$$\frac{\hat{p}_i - c_i}{\hat{p}_i} \frac{\hat{p}_i h_i''(\hat{p}_i)}{-h_i'(\hat{p}_i)} = \tilde{\mu}^f.$$

Under NCES, $(\hat{p}_i h_i''(\hat{p}_i))/(-h_i'(\hat{p}_i)) = \sigma$, so that $\tilde{\mu}^f < \sigma$. Moreover, under both NCES and NMNL demand, $\tilde{\mu}^f$ satisfies

$$\tilde{\mu}^f \left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta\tilde{\mu}^f \frac{\sum_{l \in f'} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + \sum_{l \in f'} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}, \tag{35}$$

so that $\tilde{\mu}^f > 1$.

Fix a product $i \in \mathcal{N}^f \setminus \mathcal{N}^{f\prime}$, and let $n \in f$ be the nest that contains product i. For every $x \geq c_i$, let $\tilde{G}(x)$ be the value of G when product i is priced at x and all the other products are priced according to $\hat{p}$. We showed in the previous step that $\tilde{G}(x) \underset{x \to \infty}{\longrightarrow} G(\hat{p})$. Note that, for every $x \in (c_i, \infty)$,

$$\begin{aligned} \tilde{G}'(x) = D_i \times \Bigg(& 1 - (x - c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)\frac{(x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f \sum_{j \in (n \cap \mathcal{N}^{f\prime}) \setminus \{i\}} h_j(\hat{p}_j)}{h_i(x) + \sum_{j \in (n \cap \mathcal{N}^{f\prime}) \setminus \{i\}} h_j(\hat{p}_j)} \\ & + \beta \frac{\left(h_i(x) + \sum_{j \in (n \cap \mathcal{N}^{f\prime}) \setminus \{i\}} h_j(\hat{p}_j)\right)^{\beta - 1} \left((x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f \sum_{j \in (n \cap \mathcal{N}^{f\prime}) \setminus \{i\}} h_j(\hat{p}_j)\right)}{H^{0\prime} + \left(h_i(x) + \sum_{j \in (n \cap \mathcal{N}^{f\prime}) \setminus \{i\}} h_j(\hat{p}_j)\right)^\beta + \sum_{l \in f' \setminus \{n\}} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta} \\ & + \beta\tilde{\alpha}\tilde{\mu}^f \frac{\sum_{l \in f' \setminus \{n\}} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + \left(h_i(x) + \sum_{j \in (n \cap \mathcal{N}^{f\prime}) \setminus \{i\}} h_j(\hat{p}_j)\right)^\beta + \sum_{l \in f' \setminus \{n\}} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta} \Bigg), \end{aligned} \tag{36}$$

where we have used the simplification derived in equation (18).

We argue that $\tilde{G}'(x) < 0$ for x sufficiently high. We distinguish two cases. Assume first that $n \notin f'$, i.e., $\hat{p}_j = \infty$ for every $j \in n$. Then, $\tilde{G}'(x)$ simplifies to

$$\begin{aligned} \tilde{G}'(x) = D_i \Bigg(& 1 - (x - c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)(x-c_i)\frac{-h_i'(x)}{h_i(x)} \\ & + \beta \frac{h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f \sum_{l \in f' \setminus \{n\}} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + h_i(x)^\beta + \sum_{l \in f' \setminus \{n\}} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j)\right)^\beta} \Bigg). \end{aligned} \tag{37}$$

Under NCES demand, $(x - c_i)\frac{h_i''(x)}{-h_i'(x)}$ and $(x-c_i)\frac{-h_i'(x)}{h_i(x)}$ tend to σ and $\sigma - 1$, respectively, as

x goes to infinity, whereas

$$h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) = (\sigma-1)a_i x^{\beta(1-\sigma)}\frac{x-c_i}{x}$$

tends to zero. It follows that the term in parenthesis in equation (37) tends to

$$1-\sigma+(1-\beta)(\sigma-1)+\beta\tilde{\alpha}\tilde{\mu}^f\frac{\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta}{H^{0\prime}+\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta},$$

which, using equation (35), simplifies to

$$\begin{aligned}-\beta(\sigma-1)+\tilde{\mu}^f(1-\tilde{\alpha}(1-\beta))-1 &< -\beta(\sigma-1)+\sigma(1-\tilde{\alpha}(1-\beta))-1,\\ &= \frac{1}{1-\tilde{\alpha}}\Big(-\beta\tilde{\alpha}+(1-\tilde{\alpha}(1-\beta))-(1-\tilde{\alpha})\Big),\\ &= 0.\end{aligned}$$

Hence, $\tilde{G}'(x)<0$ for high enough x.

Under NMNL demand,

$$h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) = \frac{x-c_i}{\lambda}\exp\left(\frac{\beta}{\lambda}(a_i-x)\right)\underset{x\to\infty}{\longrightarrow}0,$$

and

$$1-(x-c_i)\frac{h_i''(x)}{-h_i'(x)}+(1-\beta)(x-c_i)\frac{-h_i'(x)}{h_i(x)} = 1-\frac{\beta}{\lambda}(x-c_i)\underset{x\to\infty}{\longrightarrow}-\infty.$$

Hence, we also have that $\tilde{G}'(x)<0$ for high enough x.

Next, assume instead that $n\in f'$. Under NCES demand, the term in parenthesis in equation (36) tends to

$$1-\sigma+(1-\beta)\tilde{\alpha}\tilde{\mu}^f+\beta\tilde{\alpha}\tilde{\mu}^f\frac{\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta}{H^{0\prime}+\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}}h_j(\hat{p}_j)\right)^\beta},$$

which, using equation (35), simplifies to

$$1-\sigma+(1-\beta)\tilde{\alpha}\tilde{\mu}^f+\tilde{\mu}^f(1-\tilde{\alpha}(1-\beta))-1 = -\sigma+\tilde{\mu}^f<0,$$

implying that $\tilde{G}'(x)<0$ for x high enough.

Under NMNL demand, the term in parenthesis in equation (36) tends again to $-\infty$, so that $\tilde{G}'(x)<0$ for x high enough.

It follows that $\tilde{G}$ is strictly decreasing over some interval (x^0,∞). Therefore, $\tilde{G}(x^0) >$

$\lim_{x \to \infty} \tilde{G}(x) = G(\hat{p})$, and $\hat{p}$ does not maximize G, a contradiction. Hence, $\hat{p} \in \prod_{j \in \mathcal{N}^f} [c_j, \infty)$ maximizes G, which concludes Step 3.

Step 4: There exists a unique price vector satisfying the first-order optimality conditions. The analysis in Appendix B implies that the price vector $\hat{p} \in \prod_{j \in \mathcal{N}^f} [c_j, \infty)$ satisfies the first-order conditions if and only if there exists a $\tilde{\mu}^f$ that is such that for every $i \in \mathcal{N}^f$, $\hat{p}_i = r_i(\tilde{\mu}^f)$, where

$$r_i(x) \equiv \begin{cases} \frac{\sigma}{\sigma - x} c_i & \text{in the case of NCES,} \\ \lambda x + c_i & \text{in the case of NMNL,} \end{cases}$$

and that satisfies

$$\tilde{\mu}^f \left(1 - \tilde{\alpha}(1 - \beta)\right) = 1 + \tilde{\alpha} \beta \tilde{\mu}^f \frac{\sum_{l \in f} \left(\sum_{j \in f} h_j(r_j(\tilde{\mu}^f)) \right)^\beta}{H^{0\prime} + \sum_{l \in f} \left(\sum_{j \in f} h_j(r_j(\tilde{\mu}^f)) \right)^\beta},$$

or, equivalently,

$$\tilde{\mu}^f (1 - \tilde{\alpha}) = 1 - \tilde{\alpha} \beta \tilde{\mu}^f \frac{H^{0\prime}}{H^{0\prime} + \sum_{l \in f} \left(\sum_{j \in f} h_j(r_j(\tilde{\mu}^f)) \right)^\beta}. \tag{38}$$

As the left-hand side of equation (38) is strictly increasing, whereas the right-hand side is strictly decreasing, that equation has at most one solution. By Step 3, that equation has a solution. Hence, there exists a unique price vector satisfying the first-order conditions. □

I Competition Within and Across Nests

In this appendix, we provide a formal treatment of the extension with broad and narrow firms, introduced in the second part of Appendix B. The appendix is organized as follows. In Appendix I.1, we characterize the unique equilibrium of the oligopoly model as the solution to a nested fixed-point problem, show that the type aggregation property continues to hold, and perform comparative statics. In Appendix I.2, we make use of the type aggregation property to provide a simple conceptual framework for modeling mergers. In Appendix I.3, we develop a static analysis of the consumer surplus effects of mergers and show that all the results derived in the first part of Section 3 continue to hold. In Appendix I.4, we study the consumer surplus effects of mergers in a dynamic framework where merger opportunities arise stochastically over time and show that the result on dynamic optimality of a myopic merger policy derived in the second part of Section 3 continues to hold. In Appendix I.5, we examine whether mergers between narrow firms raise more competitive concerns than mergers between broad firms. Short proofs are provided in the main text. Longer mathematical developments are relegated to Appendix I.6.

I.1 The Oligopoly Model: Equilibrium Analysis

We know from Appendix H that for broad firms, first-order conditions are sufficient for global optimality. This implies that the behavior of broad firm $f \in \mathcal{F}^b$ with type T^f can still be described by the markup, market-share, and profit fitting-in functions $m(T^f/H)$, $S(T^f/H)$ and $\pi(T^f/H)$ defined in the first part of Appendix B.

Moreover, we know from Lemma XXI in the Online Appendix to Nocke and Schutz (2018) that first-order conditions are also sufficient for optimality for narrow firms. In the following, we use this to define firm- and nest-level fitting-in functions, establish equilibrium existence and uniqueness, and characterize the equilibrium as a nested fixed-point problem. We then perform comparative statics.

Firm-level fitting-in functions. Let f be a narrow firm operating in nest l. We know from Lemma XXI in the Online Appendix to Nocke and Schutz (2018) that the optimal prices of firm f satisfy the common ι-markup property. Let $\widetilde{\mu}^f$ be the ι-markup of firm $f \in l$. Equation (xxix) in the Online Appendix to Nocke and Schutz (2018), which characterizes firm f's optimal ι-markup as a function of H_l and H, can be rewritten as follows:[3]

$$\frac{\widetilde{\mu}^f - 1}{\widetilde{\mu}^f} = (1-\beta)\widetilde{\alpha}\frac{\sum_{j\in f} h_j}{H_l} + \beta\widetilde{\alpha}\frac{\sum_{j\in f} h_j}{H_l}\frac{H_l^{\beta}}{H}, \tag{39}$$

where, as in Appendix B, $\widetilde{\alpha}$ is equal to $(\sigma-1)/\sigma$ under NCES demand and to 1 under NMNL demand.

Define firm f's market share within nest l as

$$\widetilde{s}^f = \frac{\sum_{j\in f} h_j}{H_l}.$$

In the discrete/continuous choice micro-foundation, $\widetilde{s}^f$ is the probability that the consumer chooses a product sold by firm f conditional on having chosen nest l. The market share of nest l, which also corresponds to the probability that nest l is chosen, is given by

$$s_l = \frac{H_l^{\beta}}{H}.$$

Firm f's market share at the industry level is therefore given by $s^f \equiv \widetilde{s}^f s_l$. Market shares are still measured in value in the NCES case and in volume in the NMNL case.

[3] To see how to derive equation (39) from equation (xxix) in Nocke and Schutz (2018), note that under NCES or NMNL demand, $\Phi_l(H_l) = H_l^{\beta}$, $\Psi(H) = \log(H^0 + H)$, and $\gamma_j = \tilde{\alpha} h_j$.

Having defined market shares, we can rewrite equation (39) as follows:

$$\widetilde{\mu}^f = \frac{1}{1 - \widetilde{\alpha}\widetilde{s}^f \left(1 - \beta + \beta s_l\right)}. \tag{40}$$

Intuitively, firm f sets a high markup if it has a high market share in its nest or if its nest commands a high market share at the industry level.

Moreover, it is straightforward to show that

$$\sum_{j \in f} h_j = \left(T^f\right)^{\frac{1}{\beta}} \psi(\widetilde{\mu}^f),$$

where

$$\psi(\widetilde{\mu}^f) = \begin{cases} \left(1 - (1 - \widetilde{\alpha})\widetilde{\mu}^f\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}} & \text{under NCES demand,} \\ \mathrm{e}^{-\widetilde{\mu}^f} & \text{under NMNL demand,} \end{cases}$$

and

$$T^f = \begin{cases} \left(\sum_{j \in f} a_j c_j^{1-\sigma}\right)^{\beta} & \text{under NCES demand,} \\ \left(\sum_{j \in f} \exp \frac{a_j - c_j}{\lambda}\right)^{\beta} & \text{under NMNL demand.} \end{cases}$$

Firm f's type, T^f, has the same interpretation as in the main text: consumer surplus would be equal to $\log T^f$ if firm f were to price all of its products at marginal cost and no other firm were present in the industry.

Firm f's market share in nest l can be rewritten as follows:

$$\widetilde{s}^f = \frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l} \psi(\widetilde{\mu}^f). \tag{41}$$

It is straightforward to show that the system of equations (40)–(41) has a unique solution in $(\widetilde{\mu}^f, \widetilde{s}^f) \in (1, 1/(1 - \widetilde{\alpha})) \times \mathbb{R}_{++}$, which we denote[4]

$$\left(\widetilde{m}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}, s_l\right), \widetilde{S}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}, s_l\right)\right).$$

Clearly, $\widetilde{m}$ and $\widetilde{S}$ are smooth on $\mathbb{R}^2_{++}$. Moreover, $\widetilde{m}(\cdot,\cdot)$ is strictly increasing in both arguments, and $\widetilde{S}(\cdot)$ is strictly increasing in the first argument and strictly decreasing in the second. For some of the proofs, we will also require information on the range of $\widetilde{S}(\cdot, s_l)$ for every $s_l > 0$: we have that $\lim_{x \downarrow 0} \widetilde{S}(x, s_l) = 0$ and $\lim_{x \to \infty} \widetilde{S}(x, s_l) \geq 1$. Finally, as $1 < \widetilde{m}(\cdot,\cdot) < 1/(1 - \widetilde{\alpha})$, we have that $(1 - \beta + \beta s_l)\widetilde{S}(\cdot, s_l) < 1$ for every $s_l > 0$.

[4]In the NMNL case ($\tilde{\alpha} = 1$), the upper bound $1/(1 - \tilde{\alpha})$ is equal to ∞.

Nest-level fitting-in functions. Let $l \in \mathcal{L}^n$. The monotonicity properties of $\widetilde{S}$ and the fact that $\lim_{x \downarrow 0} \widetilde{S}(x, s_l) = 0$ and $\lim_{x \to \infty} \widetilde{S}(x, s_l) \geq 1$ imply that, for given $H > 0$, there is a unique $H_l > 0$ such that

$$\sum_{f \in l} \widetilde{S} \left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H} \right) = 1. \tag{42}$$

(See Lemma XXIII in the Online Appendix to Nocke and Schutz (2018) for a more general version of this result.) Let $H_l(H)$ be the unique solution to this equation. The monotonicity properties of $\widetilde{S}$ also imply that $H_l(\cdot)$ is strictly increasing. The function $H_l(\cdot)$ allows us to define the nest-market-share fitting-in function $\Sigma_l(\cdot)$ as

$$\Sigma_l(H) \equiv \frac{\left(H_l(H)\right)^\beta}{H}. \tag{43}$$

The argument in the proof of Lemma XXIV in the Online Appendix to Nocke and Schutz (2018) implies that $\Sigma_l(\cdot)$ is strictly decreasing, $\Sigma_l(H) > 1$ for H sufficiently close to 0, and $\lim_{H \to \infty} \Sigma_l(H) = 0$.

Equilibrium condition. The analogue of equilibrium condition (10) is:

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}^b} S\left(\frac{T^f}{H}\right) + \sum_{l \in \mathcal{L}^n} \Sigma^l(H) = 1, \tag{44}$$

i.e., the market share of the outside option, the market shares of broad firms, and the market shares of the nests of narrow firms add up to unity. The properties of the functions $S(\cdot)$ (derived in the first part of Appendix B) and $\Sigma_l(\cdot)$ (derived above) imply that equation (44) has a unique solution, which pins down the equilibrium aggregator level H^*. Hence, there is a unique equilibrium.

Profits. It is straightforward to show that the profit of narrow firm $f \in l$, π^f, is given by

$$\pi^f = \widetilde{\alpha}\beta\widetilde{\mu}^f\widetilde{s}^f s^l.$$

Thus, firm f's profit fitting-in function is given by:

$$\pi^f(H) = \widetilde{\alpha}\beta\widetilde{m}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l(H)}, \Sigma_l(H)\right)\widetilde{S}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l(H)}, \Sigma_l(H)\right)\Sigma_l(H). \tag{45}$$

The formula in the statement of Theorem III in the Online Appendix to Nocke and Schutz (2018) also implies that

$$\pi^f(H) = \left(\widetilde{m} \left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l(H)}, \Sigma_l(H) \right) - 1 \right) \frac{1}{1 + \frac{1-\beta}{\beta} \frac{1}{\Sigma_l(H)}}. \tag{46}$$

We summarize these insights in the following proposition:

Proposition 12. *A multiproduct-firm pricing game with broad and narrow firms has a unique equilibrium. The equilibrium aggregator level H^* is the unique solution of equation* (44)*. The behavior of broad firm $f \in \mathcal{F}^b$ is governed by the fitting-in functions $m(\cdot)$, $S(\cdot)$, and $\pi(\cdot)$. The behavior of narrow firm $f \in l$ is governed by the fitting-in functions $\widetilde{m}(\cdot,\cdot)$, $\widetilde{S}(\cdot,\cdot)$, $\pi^f(\cdot)$, $H_l(\cdot)$, and $\Sigma_l(\cdot)$.*

Comparative statics. The following comparative statics, which may be of independent interest, will play an important role in our merger analysis:

Proposition 13. *Consider a multiproduct-firm pricing game with broad and narrow firms. Let $l \in \mathcal{L}^n$ and $f \in \mathcal{F}_l$. In equilibrium, an increase in T^f*

(i) raises H^*,

(ii) raises H_l and s_l,

(iii) raises $H_{l'}$ and lowers $s_{l'}$ for every $l' \in \mathcal{L}^n$ such that $l' \neq l$,

(iv) raises $\widetilde{\mu}^f$, $\widetilde{s}^f$, s^f, and π^f,

(v) lowers $\widetilde{\mu}^g$, $\widetilde{s}^g$, and π^g for every $g \neq f$ in $\mathcal{F}_l$,

(vi) lowers μ^g (respectively, $\widetilde{\mu}^g$), s^g, and π^g for every $g \in \mathcal{F} \setminus \mathcal{F}_l$.

Similarly, let $f \in \mathcal{F}^b$. In equilibrium, an increase in T^f

(vii) raises H^*,

(viii) raises H_l and lowers s_l for every $l \in \mathcal{L}^n$,

(ix) raises μ^f, s^f, and π^f,

(x) lowers μ^g (respectively, $\widetilde{\mu}^g$), s^g, and π^g for every $g \in \mathcal{F} \setminus \{f\}$.

Proof. See Appendix I.6. □

I.2 Modeling Mergers

As announced in Appendix B, we confine attention to two types of mergers: broad mergers, which are such that the merger partners and the merged firm are all broad firms; and narrow mergers, which are such that the merger partners and the merged firm are all narrow firms operating in the same nest. Regardless of whether a merger is broad or narrow, the type aggregation property implies that the post-merger type T^M is a sufficient statistic for the behavior of the merged firm. We therefore continue to be agnostic on whether a given merger leads to new products being introduced (or old products being withdrawn) by the merged firm, or whether the qualities and unit costs of the merged firms' pre-existing products increase or decrease. We continue to assume that the product portfolios of non-merging firms are unaffected by the merger.

As in Section 2, a broad merger M between the firms in $\mathcal{M}$ involves synergies if the post-merger type satisfies

$$T^M > \sum_{f \in \mathcal{M}} T^f.$$

The definition of types for narrow firms in Appendix I.1 implies that a narrow merger M between the firms in $\mathcal{M}$ involves synergies if

$$(T^M)^{\frac{1}{\beta}} > \sum_{f \in \mathcal{M}} (T^f)^{\frac{1}{\beta}}.$$

I.3 Consumer Surplus Effects of Mergers: Static Analysis

As the behavior of broad firms is still driven by the fitting-in functions $S(\cdot)$, $m(\cdot)$, and $\pi(\cdot)$, the analysis in the first part of Section 3 applies to broad mergers. Specifically, Propositions 2–4 all apply to broad mergers. The objective of this subsection is to prove the analogues of those propositions for narrow mergers.

Existence of the cutoff type Consider a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$ and let T^M denote the post-merger type of the merged firm. Let H^* denote the pre-merger equilibrium aggregator level. The pre-merger equilibrium level of the nest-l aggregator is denoted H_l^*. Suppose that T^M satisfies

$$\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right) = \sum_{f \in \mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right). \tag{47}$$

Then, at H^*, nest l continues to deliver a contribution to the industry aggregator of $(H_l^*)^\beta$. As other nests are not directly affected by the merger, they continue to provide their pre-merger contribution to the industry aggregator. It follows that industry-level market shares continue to add up to unity. Therefore, H^* continues to be the equilibrium aggregator level and merger M is CS-neutral.

The fact that $(0,1) \subseteq \widetilde{S}(\mathbb{R}_{++}, s_l)$ and that $\widetilde{S}(\cdot, s_l)$ is continuous and strictly increasing implies that equation (47) has a unique solution in T^M, denoted $\widehat{T}^M(H_l^*, H^*)$.

If T^M is strictly greater than $\widehat{T}^M$, then, by Proposition 13, the post-merger equilibrium aggregator level strictly exceeds H^* and the merger is CS-increasing. The same argument implies that the merger is CS-decreasing if $T^M < \widehat{T}^M$.

We now show that $\left(\widehat{T}^M\right)^{\frac{1}{\beta}} > \sum_{f\in\mathcal{M}}(T^f)^{\frac{1}{\beta}}$. As $\widetilde{S}$ is strictly sub-additive in its first argument (see Lemma 30 in Appendix I.6), we have that

$$\widetilde{S}\left(\sum_{f\in\mathcal{M}} \frac{(T^f)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right) < \sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right).$$

Thus, at $T^M = \left(\sum_{f\in\mathcal{M}}(T^f)^{\frac{1}{\beta}}\right)^\beta$, the left-hand side of equation (47) is strictly lower than the right-hand side. As that left-hand side is strictly increasing in T^M, it follows that $\left(\widehat{T}^M\right)^{\frac{1}{\beta}} > \sum_{f\in\mathcal{M}}(T^f)^{\frac{1}{\beta}}$. In other words, a CS-nondecreasing merger must involve synergies.

We summarize these insights in the following proposition, which is the analogue of Proposition 2:

Proposition 14. *For a narrow merger among the firms in $\mathcal{M} \subseteq \mathcal{F}_l$, there exists a unique*

$$\widehat{T}^M > \left(\sum_{f\in\mathcal{M}} (T^f)^{\frac{1}{\beta}}\right)^\beta$$

such that the merger is CS-neutral if the post-merger type satisfies $T^M = \widehat{T}^M$, CS-decreasing if $T^M < \widehat{T}^M$, and CS-increasing if $T^M > \widehat{T}^M$.

Impact of the intensity of competition on the cutoff type Our goal is to prove the analogue of Proposition 3 for narrow mergers. That is, we want to show that a narrow merger requires fewer synergies to be CS-neutral if the merging firms operate in a more competitive environment. Compared to what we do in Section 3, the difference is that from the point of view of firm $f \in l$, the intensity of competition is now captured by two aggregators: H_l^* and H^*. (Lemma 29 in Appendix I.6 indicates that firm f does perceive an increase in H_l^* as competition becoming more intense as $\widetilde{m}\left(T^f/H_l^*, (H_l^*)^\beta/H^*\right)$ is decreasing in H_l^*.)

We first show that the cutoff type $\widehat{T}^M(H_l^*, H^*)$ decreases with H^*:

Proposition 15. *For a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$, the cutoff type $\widehat{T}^M(H_l^*, H^*)$ is strictly decreasing in H^*.*

Proof. See Appendix I.6. □

Next, we show that the cutoff type $\widehat{T}^M(H_l^*, H^*)$ decreases with H_l^*:

Proposition 16. *For a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$, the cutoff type $\widehat{T}^M\left(H_l^*, H^*\right)$ is strictly decreasing in H_l^*.*

Proof. See Appendix I.6. $\square$

A final thought experiment is to raise H_l^* while holding fixed $(H_l^*)^\beta/H^*$, the market share of nest l:

Proposition 17. *Consider a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$. For every $s_l \in (0,1)$,*

$$\left.\frac{\partial \widehat{T}^M(H_l^*, H^*)}{\partial H_l^*}\right|_{(H_l^*)^\beta/H^*=s_l} < 0.$$

Proof. This boils down to showing that $H_l \mapsto \widehat{T}^M\left(H_l, H_l^\beta/s_l\right)$ is strictly decreasing, which holds true by Propositions 15 and 16. $\square$

Impact of pre-merger types on the cutoff type We now prove the analogue of Proposition 4 for narrow mergers. That is, we show that narrow mergers involving larger firms require larger synergies to be CS-nondecreasing, holding fixed the pre-merger aggregator levels:

Proposition 18. *Consider a narrow merger between the firms in $\mathcal{M} = \{f, g\} \subseteq \mathcal{F}_l$, resp., $\mathcal{M}' = \{f', g'\} \subseteq \mathcal{F}_l$, where $T^f \geq T^{f'}$ and $T^g > T^{g'}$. Then, the "larger" merger $\mathcal{M}$ requires larger synergies than $\mathcal{M}'$, in the sense of a larger fractional increase in type:*

$$\frac{(\widehat{T}^M)^{\frac{1}{\beta}}}{(T^f)^{\frac{1}{\beta}} + (T^g)^{\frac{1}{\beta}}} > \frac{(\widehat{T}^{M'})^{\frac{1}{\beta}}}{(T^{f'})^{\frac{1}{\beta}} + (T^{g'})^{\frac{1}{\beta}}}.$$

This in turn implies that the larger merger requires a larger absolute increase in type:

$$(\widehat{T}^M)^{\frac{1}{\beta}} - \left((T^f)^{\frac{1}{\beta}} + (T^g)^{\frac{1}{\beta}}\right) > (\widehat{T}^{M'})^{\frac{1}{\beta}} - \left((T^{f'})^{\frac{1}{\beta}} + (T^{g'})^{\frac{1}{\beta}}\right).$$

Proof. The argument in the proof of Proposition 4 relies solely on the following properties of $S(\cdot)$: $S(\cdot)$ is smooth, positive, concave, strictly increasing, and sub-additive; $\varepsilon(\cdot)$, the elasticity of $S(\cdot)$, is strictly decreasing; $S''(\cdot)/(S'(\cdot))^2$ is strictly decreasing. As $\widetilde{S}(\cdot, s_l)$ satisfies the same properties (see Lemmas 30 and 33 in Appendix I.6) for every s_l and as the argument s_l does not vary in the statement of the proposition, the same argument can be applied to obtain the proposition. $\square$

I.4 Consumer Surplus Effects of Mergers: Dynamic Analysis

The dynamic framework is the same as in the second part of Section 3. To ensure that the static results derived in Section 3 and Appendix I.3 apply, we assume that every merger is either broad or narrow.

As in the main text, our goal is to establish the dynamic optimality of a CS-maximizing merger policy. The analysis proceeds in two main steps. First, we show that the myopically CS-maximizing merger policy maximizes discounted consumer surplus *if all feasible but not yet approved mergers are proposed in each period.* Second, we show that there exists a subgame-perfect equilibrium in which all feasible but not yet approved mergers are indeed proposed in each period. Moreover, any subgame-perfect equilibrium induces the same optimal sequence of period-by-period consumer surpluses.

We begin with the following observation:

Lemma 20. *Suppose a broad or narrow merger takes place. If the merger is CS-increasing (resp. CS-decreasing), then H^* and H_l^* increase (resp. decrease) for every nest $l \in \mathcal{L}^n$. If it is CS-neutral, then H^* and H_l^* remain constant for every nest $l \in \mathcal{L}^n$.*

Proof. Consider a broad or narrow merger M between the firms in $\mathcal{M}$. If $T^M = \widehat{T}^M$, then, by definition of the cutoff type, the merger is CS-neutral and the merger affects neither the industry aggregator nor the nest-level aggregators. If T^M increases above $\widehat{T}^M$, then the industry aggregator and the nest-level aggregators increase strictly by Proposition 13. Therefore, a CS-increasing merger raises all aggregators strictly. The same argument implies that a CS-decreasing merger lowers all aggregators strictly. □

Next, using Lemma 20 and Propositions 3, 15, and 16, we establish the sign-preserving complementarity between CS-nondecreasing (resp. CS-nonincreasing) broad or narrow mergers, extending Lemma 1:

Lemma 21. *If broad or narrow merger M_k is CS-nondecreasing in isolation, it remains CS-nondecreasing if another broad or narrow merger $M_{k'}$, $k' \neq k$, that is CS-nondecreasing in isolation takes place. If broad or narrow merger M_k is CS-decreasing in isolation, it remains CS-decreasing if another broad or narrow merger $M_{k'}$, $k' \neq k$, that is CS-decreasing in isolation takes place.*

Proof. A CS-nondecreasing merger weakly raises the industry aggregator and all nest aggregators by Lemma 20. Hence, such a merger weakly lowers the cutoff types of all other mergers by Propositions 3, 15, and 16. The same argument implies that a CS-nonincreasing merger weakly raises the cutoff types of all other mergers. □

Lemma 2 extends trivially:

Lemma 22. *Suppose that broad or narrow merger M_k is CS-nondecreasing in isolation whereas broad or narrow merger $M_{k'}$ is CS-decreasing in isolation but CS-nondecreasing once merger M_k has taken place. Then, merger M_k is CS-increasing conditional on merger $M_{k'}$ taking place.*

Combining the argument in Lemma 4 in Nocke and Whinston (2010) and Lemmas 21 and 22, we obtain the analogue of Lemma 3:

Lemma 23. *Suppose that all feasible but not yet approved mergers are proposed in each period. Then, the myopically CS-maximizing merger policy maximizes discounted consumer surplus, no matter what the realization of feasible mergers is.*

Next, we turn to the second part of our analysis. That is, we show that there always exists a subgame-perfect equilibrium in which, in each period, every feasible but not yet approved merger is proposed for approval.

We begin by showing that a CS-nondecreasing merger is privately profitable, extending the profitability result in Proposition 2:

Lemma 24. *A CS-nondecreasing broad or narrow merger is privately profitable in that it strictly raises the joint profit of the merger partners, holding fixed the market structure among outsiders.*

Proof. See Appendix I.6. □

The second step is to extend Lemma 4. That is, we want to show that a CS-nondecreasing merger is still privately profitable even if it induces (directly or indirectly) other mergers to become CS-nondecreasing, resulting in their approval. The following observation will be useful:

Lemma 25. *A CS-increasing (resp. CS-decreasing) broad or narrow merger lowers (resp. raises) the equilibrium profits of every outsider. A CS-neutral merger does not affect outsiders' profits.*

Proof. Consider a broad or narrow merger M between the firms in $\mathcal{M}$. Suppose that the post-merger type T^M is equal to $\widehat{T}^M$, so that merger M is CS-neutral. As the merger affects none of the aggregators by Lemma 20, it has no impact on the profits made by rival firms. Starting from this outcome, a CS-increasing merger is formally equivalent to increasing the post-merger type above $\widehat{T}^M$. We know from Proposition 13 that this results in lower profits for all rivals. The same argument implies that a CS-decreasing merger raises the profits of every outsider. □

We obtain the analogue of Lemma 4:

Lemma 26. *Suppose that broad or narrow merger M_k is CS-nondecreasing given current market structure whereas broad or narrow merger $M_{k'}$ is CS-decreasing but becomes CS-nondecreasing once M_k has been implemented. Then, the joint profit of the firms in $\mathcal{M}_k$ is strictly higher if both mergers take place than if none does.*

Proof. Given Lemmas 22, 24, and 25, the argument is exactly the same as in the proof of Lemma 4. □

Combining the results shown above with a backward induction argument, we obtain the main result of this subsection, extending Proposition 5:

Proposition 19. *Suppose that the antitrust authority adopts the myopically CS-maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect equilibrium. The resulting outcome maximizes discounted consumer surplus, no matter what the realized sequence of feasible mergers. Moreover, every subgame-perfect equilibrium results in the same optimal level of consumer surplus in each period.*

I.5 Comparing Broad and Narrow Mergers

Fix a vector of industry-level market shares $(s^f)_{f\in\mathcal{M}}$, where $\mathcal{M}$ is a finite set containing at least two elements. In this subsection, we study whether a merger between the firms in $\mathcal{M}$ requires more or fewer synergies to be CS-neutral, depending on whether the merger is broad or narrow. Let $\overline{s} = \sum_{f\in\mathcal{M}} s^f$ be the combined industry-level market share of the merger partners. If the merger is narrow, let $s_l \geq \overline{s}$ be the market share of the nest where the merger partners are operating.

The case of NMNL demand. Suppose first that merger $\mathcal{M}$ is a broad merger. The cutoff type that makes this merger CS-neutral satisfies

$$\overline{s} = \frac{T_b^M}{H} \exp\left(-\frac{1}{1-\overline{s}}\right),$$

where we have used equations (5) and (9). It follows that

$$\frac{T_b^M}{H} = \overline{s} \exp \frac{1}{1-\overline{s}}.$$

Similarly, the pre-merger type of firm $f \in \mathcal{M}$ satisfies

$$\frac{T_b^f}{H} = s^f \exp \frac{1}{1-s^f}.$$

The required synergy level for the broad merger is therefore given by

$$E_b = \frac{T_b^M}{\sum_{f\in\mathcal{M}} T_b^f} = \frac{\overline{s}\exp\frac{1}{1-\overline{s}}}{\sum_{f\in\mathcal{M}} s^f \exp\frac{1}{1-s^f}}.$$

Next, suppose instead that the merger is narrow. The cutoff type satisfies

$$\frac{\overline{s}}{s_l} = \frac{(T_n^M)^{\frac{1}{\beta}}}{H_l} \exp\left(-\frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right),$$

where we have used equations (26) and (27). It follows that

$$\frac{(T_n^M)^{\frac{1}{\beta}}}{H_l} = \frac{\overline{s}}{s_l} \exp \frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}.$$

Similarly, the pre-merger type of firm $f \in \mathcal{M}$ satisfies

$$\frac{(T_n^f)^{\frac{1}{\beta}}}{H_l} = \frac{s^f}{s_l} \exp \frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}}.$$

This pins down the required synergy level as

$$E_n = \frac{T_n^M}{\left(\sum_{f\in\mathcal{M}}(T_n^f)^{\frac{1}{\beta}}\right)^{\beta}} = \left(\frac{\overline{s}\exp\frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}}{\sum_{f\in\mathcal{M}} s^f \exp\frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}}}\right)^{\beta}.$$

Comparing E_b and E_n, we obtain:

Lemma 27. *Consider two equivalent broad and narrow mergers. Let $\overline{s}$ be the combined pre-merger industry-level market shares of the merger partners and s_l the pre-merger market share of the narrow merger's nest. Suppose that demand is of the NMNL type and that $s^f/s_l \leq 3/4$ for every f.*

There exists a threshold $\widehat{s}_l \in (\overline{s}, 1)$ such that the broad merger requires fewer synergies than the narrow one, $E_b < E_n$, if $s_l < \widehat{s}_l$, whereas the opposite is true if $s_l > \widehat{s}_l$.

Proof. See Appendix I.6. □

Note that the condition that $s^f/s_l \leq 3/4$ for every f is automatically satisfied if the merger partners are symmetric, implying Proposition 11 in the case of NMNL demand.

The case of NCES demand Suppose the merger partners are symmetric: $s^f = \overline{s}/N$ for every f, where N is the number of merging firms. If merger $\mathcal{M}$ is a broad merger, then the cutoff type satisfies

$$\overline{s} = \frac{T_b^M}{H}\left(1 - \frac{1-\alpha}{1-\alpha\overline{s}}\right)^{\frac{\alpha}{1-\alpha}},$$

i.e.,

$$\frac{T_b^M}{H} = \overline{s}\left(\frac{1}{\alpha}\frac{1-\alpha\overline{s}}{1-\overline{s}}\right)^{\frac{\alpha}{1-\alpha}}.$$

Similarly, the pre-merger type of every merger partner satisfies

$$\frac{T_b}{H} = \frac{\overline{s}}{N}\left(\frac{1}{\alpha}\frac{1-\alpha\frac{\overline{s}}{N}}{1-\frac{\overline{s}}{N}}\right)^{\frac{\alpha}{1-\alpha}}.$$

The required synergy level for the broad merger is therefore given by

$$E_b = \frac{T_b^M}{NT_b} = \left(\frac{1-\alpha\overline{s}}{1-\overline{s}}\frac{1-\frac{\overline{s}}{N}}{1-\alpha\frac{\overline{s}}{N}}\right)^{\frac{\alpha}{1-\alpha}}.$$

Using the fact that $\alpha = \widetilde{\alpha}\beta/(1-\widetilde{\alpha}(1-\beta))$ and simplifying, we obtain:

$$E_b = \left(\frac{1-\widetilde{\alpha}(1-\beta+\beta\overline{s})}{1-\overline{s}}\frac{1-\frac{\overline{s}}{N}}{1-\widetilde{\alpha}\left(1-\beta+\beta\frac{\overline{s}}{N}\right)}\right)^{\frac{\widetilde{\alpha}\beta}{1-\widetilde{\alpha}}}.$$

Next, suppose instead that the merger is narrow. The cutoff type satisfies

$$\frac{\overline{s}}{s_l} = \frac{(T_n^M)^{\frac{1}{\beta}}}{H_l}\left(1-\frac{1-\widetilde{\alpha}}{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}},$$

i.e.,

$$\frac{(T_n^M)^{\frac{1}{\beta}}}{H_l} = \frac{\overline{s}}{s_l}\left(\frac{1}{\widetilde{\alpha}}\frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}}.$$

Similarly, the pre-merger type of every merger partner satisfies

$$\frac{(T_n)^{\frac{1}{\beta}}}{H_l} = \frac{\overline{s}}{Ns_l}\left(\frac{1}{\widetilde{\alpha}}\frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}}.$$

This pins down the required synergy level for the narrow merger as

$$E_n = \frac{\widetilde{T}^M}{\left(NT_n^{\frac{1}{\beta}}\right)^{\beta}} = \left(\frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\frac{1-(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}\right)^{\frac{\widetilde{\alpha}\beta}{1-\widetilde{\alpha}}}.$$

Comparing E_b and E_n, we obtain Proposition 11 in the case of NCES demand:

Lemma 28. *Consider two equivalent broad and narrow mergers involving symmetric firms. Let $\overline{s}$ be the combined pre-merger industry-level market shares of the merger partners and s_l the pre-merger market share of the narrow merger's nest. Suppose that demand is of the NCES type.*

There exists a threshold $\widehat{s}_l \in (\overline{s}, 1)$ such that the broad merger requires fewer synergies than the narrow one, $E_b < E_n$, if $s_l < \widehat{s}_l$, whereas the opposite is true if $s_l > \widehat{s}_l$.

Proof. See Appendix I.6. □

I.6 Technical Lemmas and Proofs

We begin by stating and proving a series of technical lemmas on the properties of fitting-in functions. The following lemma will be used in the proof of Proposition 13:

Lemma 29. *The mapping* $H_l \mapsto \widetilde{m}\left((T^f)^{\frac{1}{\beta}}/H_l, H_l^\beta/H\right)$ *is strictly decreasing.*

Proof. Note that[5]

$$\frac{d}{dH_l}\widetilde{m}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) = \frac{1}{H_l}\left(-\frac{(T^f)^{\frac{1}{\beta}}}{H_l}\partial_1\widetilde{m}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) + \beta\frac{H_l^\beta}{H}\partial_2\widetilde{m}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right).$$

Thus, all we need to do is show that $\beta y\partial_2\widetilde{m}(x,y) - x\partial_1\widetilde{m}(x,y) < 0$. Totally differentiating equations (40) and (41), we obtain:

$$\beta y\partial_2\widetilde{m}(x,y) - x\partial_1\widetilde{m}(x,y) = \frac{-\widetilde{\alpha}s(1-\beta)\left(1-(1-\beta+\beta y)\,s\right)(1+\beta y)}{\left(1-\widetilde{\alpha}\left(1-\beta+\beta y\right)s\right)\left(1-(1-\beta+\beta y)\,s+\widetilde{\alpha}\left(1-\beta+\beta y\right)^2 s^2\right)},$$

where $s = \widetilde{S}(x,y)$. The above expression is clearly negative as $(1-\beta+\beta y)\widetilde{S}(x,y) < 1$. □

The following lemma will be used in the proof of Propositions 14 and 18:

Lemma 30. $\widetilde{S}(\cdot, s_l)$ *is strictly concave. Therefore, it is strictly sub-additive.*

Proof. Applying the implicit function theorem to equations (40) and (41), we obtain:

$$\partial_1\widetilde{S}(x,y) = \frac{\widetilde{S}(x,y)\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)\right)\left(1-\widetilde{\alpha}(1-\beta+\beta y)\widetilde{S}(x,y)\right)}{x\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x,y)^2\right)}. \tag{48}$$

Differentiating equation (48) with respect to x and plugging in the value of $\partial_1\widetilde{S}(x,y)$ from that same equation yields:

$$\partial^2_{11}\widetilde{S}(x,y) = \frac{\widetilde{\alpha}s^2(1-\beta+\beta y)\left(2-(1-\beta+\beta y)s\right)\left(1-(1-\beta+\beta y)s\right)\left(1-\widetilde{\alpha}(1-\beta+\beta y)s\right)}{-x^2\left(1-(1-\beta+\beta y)s+\widetilde{\alpha}(1-\beta+\beta y)^2 s^2\right)^3},$$

where $s = \widetilde{S}(x,y)$. As $(1-\beta+\beta y)\widetilde{S}(x,y) < 1$, it follows that $\partial^2_{11}\widetilde{S}(x,y) < 0$ and that $\widetilde{S}(\cdot,y)$ is strictly sub-additive. □

The following lemma will be used in the proof of Proposition 15:

Lemma 31. *For every* (x,y), $\partial_2\widetilde{S}(x,y) = -\zeta\left(\widetilde{S}(x,y)\right)$, *where*

$$\zeta(s) \equiv \frac{\widetilde{\alpha}\beta s^2}{1-(1-\beta+\beta y)s+\widetilde{\alpha}(1-\beta+\beta y)^2 s^2}.$$

[5]Notation: $\partial_k f$ is the partial derivative of f with respect to its k^{th} argument; $\partial^2_{kl}f$ is the (cross-)partial derivative with respect to arguments k and l.

Moreover, ζ is strictly super-additive on the interval $(0, 1/(1-\beta+\beta y))$ *for every* $(\widetilde{\alpha}, \beta, y) \in (0,1] \times (0,1)^2$.

Proof. The fact that $\partial_2 \widetilde{S}(x,y) = -\zeta\left(\widetilde{S}(x,y)\right)$ follows immediately by applying the implicit function theorem to equations (40) and (41). Note that establishing the strict super-additivity of ζ on $(0, 1/(1-\beta+\beta y))$ is equivalent to establishing the strict super-additivity of

$$\tilde{\zeta}(s) = \frac{s^2}{1 - s + \widetilde{\alpha} s^2}$$

on $(0,1)$ for every $\bar{s} \in (0,1)$, which we undertake next.

Note that $\tilde{\zeta}''(s)$ has the same sign as $P(s) \equiv 1 - \widetilde{\alpha}(3-s)s^2$. The polynomial P is strictly decreasing on $[0,1]$. Hence, if $\widetilde{\alpha} \leq 1/2$, then $P(s) > 0$ for every $s \in [0,1)$ and $\tilde{\zeta}$ is strictly convex on $(0,1)$. If instead $\widetilde{\alpha} > 1/2$, then $P(1) < 0$ and there exists a unique $\hat{s}(\widetilde{\alpha}) \in (0,1)$ such that $P(\hat{s}(\widetilde{\alpha})) = 0$. It is easy to check that $P(1/2) > 0$, so that $\hat{s}(\widetilde{\alpha}) > 1/2$ for every $\widetilde{\alpha} > 1/2$.

To sum up, if $\widetilde{\alpha} \leq 1/2$, then $\tilde{\zeta}$ is strictly convex and hence strictly super-additive on $(0,1)$. If instead $\widetilde{\alpha} > 1/2$, which we assume in the following, then $\tilde{\zeta}$ is strictly convex on $(0, \hat{s}(\widetilde{\alpha}))$ and strictly concave on $(\hat{s}(\widetilde{\alpha}), 1)$. We want to show that, for every $n \geq 2$, $s \in (0,1)$, and $(s_i)_{1 \leq i \leq n} \in (0,s)^n$ such that $\sum_{i=1}^n s_i = s$, $\sum_{i=1}^n \tilde{\zeta}(s_i) < \tilde{\zeta}(s)$.

Let $s \in (0,1)$. Define

$$\mathcal{S}^n = \left\{ (s_i)_{1 \leq i \leq n} \in [0,s]^n : \sum_{i=1}^n s_i = s \right\}, \quad \forall n \geq 1,$$

$$\text{and } \mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}^n,$$

and consider the following maximization problem:

$$\max_{(s_i)_{1 \leq i \leq n} \in \mathcal{S}} \sum_{i=1}^n \tilde{\zeta}(s_i). \tag{49}$$

We need to show that $(s_i)_{1 \leq i \leq n} \in \mathcal{S}$ solves the above maximization problem if and only if $s_i = s$ for some i.

An induction argument similar to the one in the proof of Proposition 7 implies that, for every $(s_i)_{1 \leq i \leq n} \in \mathcal{S}$ with at least three strictly positive components, there exists $(s_i')_{1 \leq i \leq 2} \in \mathcal{S}^2$ such that $\sum_{i=1}^n \tilde{\zeta}(s_i) < \tilde{\zeta}(s_1') + \tilde{\zeta}(s_2')$. As, by continuity and compactness, the maximization problem

$$\max_{(s_1,s_2) \in \mathcal{S}^2} \tilde{\zeta}(s_1) + \tilde{\zeta}(s_2) \tag{50}$$

has a solution, we can conclude that maximization problem (49) has a solution, and that any solution has at most two strictly positive components. All we need to do now is show that maximization problem (50) has exactly two solutions: $(s,0)$ and $(0,s)$. This boils down to showing that $\arg\max_{\lambda\in[0,s]}\chi(\lambda) = \{0, s\}$, where $\chi(\lambda) = \tilde{\zeta}(\lambda) + \tilde{\zeta}(s-\lambda)$. Routine but tedious calculations show that this is the case, as χ is strictly decreasing on $(0, s/2)$ and strictly increasing on $(s/2, s)$. $\square$

The following lemma will be used in the proof of Proposition 16:

Lemma 32. *For every (x, y), $-x\partial_1\widetilde{S}(x, y) + \beta y\partial_2\widetilde{S}(x, y) = \phi\left(\widetilde{S}(x, y)\right)$, where*

$$\phi(s) \equiv -s + \frac{1-\beta}{\beta}(1+\beta y)\zeta(s),$$

where ζ was defined in Lemma 31. Moreover, ϕ is strictly super-additive on the interval $(0, 1/(1-\beta+\beta y))$ for every $(\widetilde{\alpha}, \beta, y) \in (0, 1] \times (0, 1)^2$.

Proof. The fact that $-x\partial_1\widetilde{S}(x, y) + \beta y\partial_2\widetilde{S}(x, y) = \phi\left(\widetilde{S}(x, y)\right)$ follows immediately by applying the implicit function theorem to equations (40) and (41). The strict super-additivity of ϕ follows as ϕ is the sum of a linear function and a strictly super-additive function (recall Lemma 31). $\square$

The following lemma will be used in the proof of Proposition 18

Lemma 33. *For every (x, y), let $\widetilde{\varepsilon}(x, y) = x\partial_1\widetilde{S}(x, y)/\widetilde{S}(x, y)$. Then, $\widetilde{\varepsilon}(\cdot, y)$ is strictly decreasing. Moreover, $\partial_{11}\widetilde{S}(\cdot, y)/\left(\partial_1\widetilde{S}(\cdot, y)\right)^2$ is strictly decreasing.*

Proof. Equation (48) implies that

$$\widetilde{\varepsilon}(x, y) = \frac{\left(1-(1-\beta+\beta y)\widetilde{S}(x, y)\right)\left(1-\widetilde{\alpha}(1-\beta+\beta y)\widetilde{S}(x, y)\right)}{1-(1-\beta+\beta y)\widetilde{S}(x, y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x, y)^2}.$$

Differentiating this with respect to x yields:

$$\partial_1\widetilde{\varepsilon}(x, y) = \frac{-\widetilde{\alpha}(1-\beta+\beta y)\left(1-\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x, y)^2\right)}{\left(1-(1-\beta+\beta y)\widetilde{S}(x, y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x, y)^2\right)^2}\partial_1\widetilde{S}(x, y),$$

which is strictly negative.

Using equation (48), we also obtain an expression for $\partial_{11}\widetilde{S}(x, y)/\left(\partial_1\widetilde{S}(x, y)\right)^2$:

$$\begin{aligned}\frac{\partial_{11}^2\widetilde{S}(x, y)}{(\partial_1\widetilde{S}(x, y))^2} = -\widetilde{\alpha}\,(1-\beta+\beta y)\left(2-(1-\beta+\beta y)\widetilde{S}(x, y)\right)/\left(\left(1-(1-\beta+\beta y)\widetilde{S}(x, y)\right)\right.\\ \left.\times\left(1-\widetilde{\alpha}(1-\beta+\beta y)\widetilde{S}(x, y)\right)\left(1-(1-\beta+\beta y)\widetilde{S}(x, y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x, y)^2\right)\right),\end{aligned}$$

which is strictly negative. $\square$

Proof of Proposition 13. Let $l \in \mathcal{L}^n$ and $f \in \mathcal{F}_l$. In this proof, we make explicit the dependence of the fitting-in functions $H_{l'}(\cdot)$ and $\Sigma_{l'}(\cdot)$ on T^f by writing $H_{l'}(\cdot, T^f)$ and $\Sigma_{l'}(\cdot, T^f)$ for every $l' \in \mathcal{L}^n$. The equilibrium values of all variables are denoted with a star superscript.

As $\widetilde{S}$ is increasing in the first argument and decreasing in the second argument, an increase in T^f raises the left-hand side of equation (42). As that left-hand side is strictly decreasing in H_l, it follows that $H_l(H, T^f)$ is strictly increasing in T^f. Hence, $\Sigma_l(H, T^f)$ is strictly increasing in T^f. By contrast, the increase in T^f leaves the left-hand side of equation (42) unaffected for nests $l' \neq l$. It follows that for $l' \neq l$, $H_{l'}(H, T^f)$ and $\Sigma_{l'}(H, T^f)$ are both constant in T^f. This immediately implies that the left-hand side of equation (44) increases as T^f increases. As that left-hand side is strictly decreasing in H, it follows that H^* is strictly increasing in T^f, which proves part (i) of the proposition.

Let $l' \in \mathcal{L}^n$ such that $l' \neq l$. As $\Sigma_{l'}(H, T^f)$ decreases with H but is constant in T^f, we immediately have that $s^*_{l'}$ is strictly decreasing in T^f. Similarly, as $H_{l'}(H, T^f)$ increases with H but does not depend on T^f, we also have that $H^*_{l'}$ is strictly increasing in T^f. This proves part (iii) of the proposition.

As the fitting-in functions $m(\cdot)$, $S(\cdot)$, and $\pi(\cdot)$ are all decreasing, and as H^* is increasing in T^f, it follows that μ^{*g}, s^{*g}, and π^{*g} are all strictly decreasing in T^f for every $g \in \mathcal{F}^b$. Next, let $l' \in \mathcal{L}^n \setminus \{l\}$ and $g \in l'$. Firm g's equilibrium ι-markup $\widetilde{\mu}^{*g}$ is given by $\widetilde{m}((T^g)^{\frac{1}{\beta}}/H^*_{l'}, s^*_{l'})$. As $\widetilde{m}$ is increasing in both arguments, and as both arguments are decreasing in T^f, it follows that $\widetilde{\mu}^{*g}$ is strictly decreasing in T^f. Combining this with the fact that $s^*_{l'}$ is decreasing in T^f and using equation (46) allows us to conclude that π^{*g} is strictly decreasing in T^f. Assume for a contradiction that $s^{*g}(T^{f\prime}) \geq s^{*g}(T^f)$ for some $T^{f\prime} > T^f$. As $s^*_{l'}(T^{f\prime}) < s^*_{l'}(T^f)$, it must be the case that $\widetilde{s}^{*g}(T^{f\prime}) > \widetilde{s}^{*g}(T^f)$. Using equation (40), this implies that $\widetilde{\mu}^{*g}(T^{f\prime}) > \widetilde{\mu}^{*g}(T^f)$, which contradicts the fact that $\widetilde{\mu}^{*g}$ is strictly decreasing in T^f. Hence, s^{*g} is strictly decreasing in T^f. This proves part (vi) of the proposition.

We have shown above that $s^*_{l'}$ is strictly decreasing in T^f for every $l' \in \mathcal{L}^n \setminus \{l\}$ and that s^{*g} is strictly decreasing in T^f for every $g \in \mathcal{F}^b$. As the equilibrium market share of the outside option, H^0/H^*, is non-increasing in T^f and as market shares must add up to unity, it follows that s^*_l is strictly increasing in T^f. Moreover, as $H_l(H, T^f)$ is strictly increasing in both of its arguments, we also have that H^*_l is strictly increasing in T^f, which proves part (ii) of the proposition.

Let $g \in \mathcal{F}_l$ such that $g \neq f$. Firm g's equilibrium market share in nest l, $\widetilde{s}^{*g}$, is given by $\widetilde{S}((T^g)^{\frac{1}{\beta}}/H^*_l, s^*_l)$. As $\widetilde{S}$ is strictly increasing in its first argument and strictly decreasing in its second argument, and as $(T^g)^{\frac{1}{\beta}}/H^*_l$ and s^*_l are respectively strictly decreasing and strictly increasing in T^f, it follows that $\widetilde{s}^{*g}$ is strictly decreasing in T^f. Moreover, as market shares within nest l add up to unity, we have that $\widetilde{s}^{*f}$ is increasing in T^f. Combining this with the fact that s^*_l is increasing in T^f and using equations (40) and (46) allows us to conclude that s^{*f}, $\widetilde{\mu}^{*f}$, and π^{*f} are all strictly increasing in T^f, which proves part (iv) of the proposition.

Finally, we prove part (v) of the proposition. Let $g \in l$ such that $g \neq f$. We have already shown that $\widetilde{s}^{*g}$ is strictly decreasing in T^f. Firm g's equilibrium ι-markup, μ^{*g}, is given by

$\widetilde{m}\left((T^g)^{\frac{1}{\beta}}/H_l^*, (H_l^*)^\beta/H^*\right)$. This expression is strictly decreasing in H_l^* (by Lemma 29) and H^* (as $\widetilde{m}$ is strictly increasing in its second argument). As H_l^* and H^* are both increasing in T^f, it follows that $\widetilde{\mu}^{*g}$ is strictly decreasing in T^f. To show that π^{*g} is strictly decreasing in T^f, define

$$H_{-l}^0(T^f) = H^0 + \sum_{l' \in \mathcal{L}^n \setminus \{l\}} \left(H_{l'}^*(T^f)\right)^\beta + \sum_{f' \in \mathcal{F}^b} H^*(T^f) S\left(\frac{T^{f'}}{H^*}\right)$$

and

$$H_{-g}^0(T^f) = \sum_{f' \in \mathcal{F}_l \setminus \{g\}} \left(T^{f'}\right)^{\frac{1}{\beta}} \psi\left(\widetilde{\mu}^{*f'}(T^f)\right).$$

We have shown that the second term in the definition of $H_{-l}^0(T^f)$, which represents the contribution of nests $l' \in \mathcal{L}^n \setminus \{l\}$ to the industry aggregator, increases with T^f. Similarly, the third term in the definition of $H_{-l}^0(T^f)$, which represents the contribution of broad firms to the industry aggregator, is also increasing in T^f as broad firms lower their ι-markups as T^f increases. Hence, $H_{-n}^0(T^f)$ is strictly increasing in T^f.

Next, we argue that $H_{-g}^0(T^f)$, the contribution of firm g's rivals to the nest aggregator H_l, also increases with T^f. Clearly, $(T^{f'})^{\frac{1}{\beta}}\psi\left(\widetilde{\mu}^{*f'}(T^f)\right)$ is increasing in T^f for every $f' \neq f$ as $\widetilde{\mu}^{*f'}$ is strictly decreasing in T^f. Moreover, we also have that $(T^f)^{\frac{1}{\beta}}\psi\left(\widetilde{\mu}^{*f}(T^f)\right) = H_l^*(T^f)\widetilde{s}^{*f}(T^f)$ increases with T^f as H_l^* and $\widetilde{s}^{*f}$ are both strictly increasing in T^f. Therefore, $H_{-g}^0(T^f)$ is strictly increasing in T^f.

The profit of firm g when it sets a ι-markup of $\widetilde{\mu}^g$ is given by:

$$\Pi(\widetilde{\mu}^g, T^f) = \tilde{\alpha}\beta(T^g)^{\frac{1}{\beta}}\widetilde{\mu}^g\psi(\widetilde{\mu}^g)\frac{\left(H_{-g}^0(T^f) + (T^g)^{\frac{1}{\beta}}\psi(\widetilde{\mu}^g)\right)^{\beta-1}}{\left(H_{-g}^0(T^f) + (T^g)^{\frac{1}{\beta}}\psi(\widetilde{\mu}^g)\right)^\beta + H_{-l}^0(T^f)},$$

which is strictly decreasing in $H_{-g}^0(T^f)$ and $H_{-l}^0(T^f)$ and thus strictly decreasing in T^f. Combining this with a standard revealed profitability argument, we can conclude that firm g's equilibrium profits fall as T^f rises. This proves part (v) of the proposition.

The proof of parts (vii)–(x) is analogous and therefore omitted. □

Proof of Proposition 15. Fix H_l and let

$$\xi(T^M, H) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) - \sum_{f \in \mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right).$$

Note that $\xi(T^M, H) = 0$ if and only if $T^M = \widehat{T}^M(H_l, H)$. Moreover, $\partial_1 \xi > 0$. All we need to do is show that $\partial_2 \xi(T^M, H) > 0$ whenever $\xi(T^M, H) = 0$. The proposition will then follow by the implicit function theorem.

Let (T^M, H) such that $\xi(T^M, H) = 0$. Note that

$$\begin{aligned}
\partial_2 \xi(T^M, H) &= \frac{H_l^\beta}{H^2}\left(\sum_{f\in\mathcal{M}} \partial_2 \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) - \partial_2 \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right), \\
&= \frac{H_l^\beta}{H^2}\left(\zeta\left(\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right) - \sum_{f\in\mathcal{M}} \zeta\left(\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right)\right), \\
&> 0,
\end{aligned}$$

where we have used the function ζ defined in Lemma 31 and the inequality follows from the fact that ζ is super-additive and

$$\sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)$$

as $\xi(T^M, H) = 0$. □

Proof of Proposition 16. The approach is similar to the one used in the proof of Proposition 15. Fix H, and define

$$\xi(T^M, H_l) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) - \sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right).$$

Note that $\xi(T^M, H_l) = 0$ if and only if $T^M = \widehat{T}^M(H_l, H)$, and that $\partial_1 \xi > 0$. We need to show that $\partial_2 \xi(T^M, H_l) > 0$ whenever $\xi(T^M, H_l) = 0$.

Let (T^M, H_l) such that $\xi(T^M, H_l) = 0$. Note that

$$\begin{aligned}
\partial_2 \xi(T^M, H_l) = \frac{1}{H_l}\Bigg(&-\frac{(T^M)^{\frac{1}{\beta}}}{H_l}\partial_1 \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) + \beta\frac{H_l^\beta}{H}\partial_2 \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) \\
&- \sum_{f\in\mathcal{M}}\left(-\frac{(T^f)^{\frac{1}{\beta}}}{H_l}\partial_1 \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) + \beta\frac{H_l^\beta}{H}\partial_2 \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right)\Bigg).
\end{aligned}$$

Hence, using the function ϕ defined in Lemma 32,

$$\partial_2 \xi(T^M, H_l) = \frac{1}{H_l}\left(\phi\left(\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right) - \sum_{f\in\mathcal{M}} \phi\left(\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right)\right) > 0,$$

where the inequality follows from the fact that ϕ is super-additive and

$$\sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right).$$

□

Proof of Lemma 24. The fact that a CS-nondecreasing broad merger is privately profitable follows immediately from the argument in the proof of Proposition 2.

Consider next a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$. We first show that the merger is profitable if it is CS-neutral, i.e., if $T^M = \widehat{T}^M$. As $T^M > T^f$ for every $f \in \mathcal{M}$, and as the merger affects neither H_l^* nor H^*, we have that firm M's ι-markup strictly exceeds the pre-merger ι-markup of every merger partner, i.e., $\widetilde{\mu}^M > \widetilde{\mu}^f$ for every f in $\mathcal{M}$. Moreover, as the merger affects neither H_l^* nor H^*, it does not affect s_l. Finally, by definition of $\widehat{T}^M$, we also have that the market share of firm M in nest l is equal to the sum of the pre-merger market shares of the merger partners, i.e., $\widetilde{s}^M = \sum_{f \in \mathcal{M}} \widetilde{s}^f$. It follows that

$$\begin{aligned}
\pi^M &= \widetilde{\alpha}\beta\widetilde{\mu}^M\widetilde{s}^M s_l, \\
&= \widetilde{\alpha}\beta\left(\sum_{f\in\mathcal{M}} \widetilde{\mu}^M\widetilde{s}^f\right) s_l, \\
&> \widetilde{\alpha}\beta\left(\sum_{f\in\mathcal{M}} \widetilde{\mu}^f\widetilde{s}^f\right) s_l, \\
&= \sum_{f\in\mathcal{M}} \pi^f.
\end{aligned}$$

The merger is therefore profitable. Moreover, by Proposition 13, firm M makes even more profits if its type is $T^M > \widehat{T}^M$. It follows that a CS-nondecreasing merger is profitable. $\square$

Proof of Lemma 27. The proof proceeds in several steps. We first show that $E_b < E_n$ when $s_l = \overline{s}$. Next, we show that $E_b > E_n$ when $s_l = 1$. Finally, we show that E_b/E_n is strictly increasing in s_l.

Assume that $s_l = \overline{s}$ and define

$$\Psi(\beta) = \beta \log \sum_{f\in\mathcal{M}} \widetilde{s}^f \exp \frac{1}{1-(1-\beta+\beta\overline{s})\widetilde{s}^f},$$

where $\widetilde{s}^f = s^f/\overline{s}$. Note that $E_n/E_b = \exp\left(\Psi(1) - \Psi(\beta)\right)$, so all we need to do is show that Ψ is increasing. Differentiating Ψ with respect to β yields:

$$\Psi'(\beta) = \frac{\Psi(\beta)}{\beta} - \frac{\sum_{f\in\mathcal{M}} \frac{\beta\widetilde{s}^f(1-\overline{s})}{\left(1-(1-\beta+\beta\overline{s})\widetilde{s}^f\right)^2}\widetilde{s}^f \exp\frac{1}{1-(1-\beta+\beta\overline{s})\widetilde{s}^f}}{\sum_{f\in\mathcal{M}} \widetilde{s}^f \exp\frac{1}{1-(1-\beta+\beta\overline{s})\widetilde{s}^f}}.$$

As $\Psi(\beta)/\beta > 1$, a sufficient condition for $\Psi'(\beta)$ to be positive is that

$$\frac{\beta\widetilde{s}^f(1-\overline{s})}{\left(1-(1-\beta+\beta\overline{s})\widetilde{s}^f\right)^2} \leq 1$$

for every f. This condition is equivalent to $\widetilde{s}^f$ being no greater than $\check{s}(\overline{s},\beta)$ for every f, where

$$\check{s}(\overline{s},\beta) = \frac{2-\beta(1-\overline{s}) - \sqrt{\beta(1-\overline{s})\left(4-3\beta(1-\overline{s})\right)}}{2\left(1-\beta(1-\overline{s})\right)^2}.$$

We find that $\check{s}$ achieves its global minimum at $\overline{s} = 5/16$ and $\beta = 16/33$. The minimized value of $\check{s}$ is $3/4$. Hence, $\Psi'(\beta) > 0$ provided $\widetilde{s}^f \leq 3/4$ for every f. This concludes the first step of the proof.

Next, assume that $s_l = 1$. Then, E_n simplifies to

$$E_n = \left(\frac{\overline{s}\exp\frac{1}{1-\overline{s}}}{\sum_{f\in\mathcal{M}} s^f \exp\frac{1}{1-s^f}}\right)^\beta = E_b^\beta.$$

As $E_b > 1$ and $\beta < 1$, it follows that $E_n < E_b$, which concludes the second step of the proof.

For the third step of the proof, note that for every $s_l \in (\overline{s}, 1)$, E_n can be rewritten as

$$E_n = \left(\sum_{f\in\mathcal{M}} \frac{s^f}{\overline{s}} \exp\left(\frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}} - \frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right)\right)^{-\beta}.$$

Moreover,

$$\frac{\partial}{\partial s_l}\left(\frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}} - \frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right) = \frac{1-\beta}{s_l^2}\left(\frac{\overline{s}}{\left(1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}\right)^2} - \frac{s^f}{\left(1-(1-\beta+\beta s_l)\frac{s^f}{s_l}\right)^2}\right),$$

which is strictly positive as $\overline{s} > s^f$. It follows that E_n is decreasing in s_l. As E_b does not depend on s_l, this implies that E_b/E_n is increasing in s_l. □

Proof of Lemma 28. Note that $E_b/E_n = (\Psi(s_l,\overline{s}))^{\frac{\widetilde{\alpha}\beta}{1-\widetilde{\alpha}}}$, where

$$\Psi(s_l,\overline{s}) \equiv \frac{1-\widetilde{\alpha}(1-\beta+\beta\overline{s})}{1-\overline{s}} \frac{1-\frac{\overline{s}}{N}}{1-\widetilde{\alpha}\left(1-\beta+\beta\frac{\overline{s}}{N}\right)} \frac{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}} \frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{N s_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{N s_l}}.$$

Note that $E_b/E_n > 1$ if $\Psi(s_l,\overline{s}) > 1$, and $E_b/E_n < 1$ if $\Psi(s_l,\overline{s}) < 1$. Routine but tedious calculations show that $\Psi(\overline{s},\overline{s}) < 1$, $\Psi(1,\overline{s}) > 1$, and $\partial\Psi/\partial s_l > 0$, which proves the lemma. □

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