Oligopoly, Complementarities, and Transformed Potentials*

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Abstract

We adopt a potential games approach to study multiproduct-firm pricing games where products can be local complements or substitutes. We show that any such game based on an IIA demand system admits an ordinal potential, giving rise to a simple proof of equilibrium existence. We introduce the concept of transformed potential, and characterize the class of demand systems that give rise to multiproduct-firm pricing games admitting such a potential, as well as the associated transformation functions. The resulting demand systems allow for substitutability or complementarity patterns that go beyond IIA, and can resemble those induced by "one-stop shopping" behavior.

Keywords: Multiproduct firms, potential game, oligopoly pricing, IIA demand, complementary goods

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1 Introduction

Multiproduct firms selling horizontally differentiated goods are ubiquitous and many markets are dominated by a small number of firms wielding market power. This is reflected in the empirical industrial organization literature, where multiproduct-firm oligopoly features prominently (e.g., Berry, Levinsohn, and Pakes, 1995; Nevo, 2001; Miller and Weinberg, 2017).

The theoretical analysis of such markets, however, is hampered by a number of technical difficulties, such as payoff functions failing to be quasi-concave (Spady, 1984; Hanson and Martin, 1996) and/or (log-)supermodular (e.g., Whinston, 2007, footnote 8). In Nocke and Schutz (2018), those difficulties are circumvented by means of an aggregative games approach, but at the cost of imposing some technical regularity conditions as well as, more substantially, the restriction to substitute products.

In this paper, we propose a different approach to the proof of equilibrium existence in multiproduct-firm oligopoly, namely one based on the theory of potential games. A normal-form game is said to admit a *potential* if there exists a function, called the potential function, such that whenever a player changes her action, the variation in her payoff is equal to the variation in the potential function (Monderer and Shapley, 1996b). Under the weaker concept of an ordinal potential, all that is required is that the variation in the deviating player's payoff has the same sign as the variation in the ordinal potential function. In such games, equilibrium existence can be established without solving a multidimensional fixed point problem (as in the best-response approach) or a nested fixed point problem (as in the aggregative games approach): An action profile that globally maximizes the (ordinal) potential is a Nash equilibrium.

In the first part of the paper (Section 2), we study multiproduct-firm pricing games based on demand systems satisfying the independence of irrelevant alternatives (IIA) property. We show that any such game admits an ordinal potential.[1] Based on this insight, we then prove existence of equilibrium under minimal assumptions on demand by showing that the ordinal potential function has a global maximizer. Importantly, the demand framework does not impose that products be substitutes, but instead allows products to be local complements or substitutes, depending on the level of prices.

Our results in the first part of the paper raise the question whether there may be other demand systems such that the induced multiproduct-firm pricing game admits an ordinal potential. Unfortunately, there is no known way of providing a complete solution to this problem: While Monderer and Shapley (1996b) provide a cross-partial derivatives test that

[1] Thus, such games are both aggregative and ordinal potential games. Connections between (variants of) aggregative games and (variants of) potential games have been explored in earlier work by Dubey, Haimanko, and Zapechelnyuk (2006) and Jensen (2010).

allows to verify easily whether a given game admits a potential, no such test is known for the weaker concept of ordinal potential.[2] The starting point of our approach to this question is the observation that the multiproduct-firm pricing game studied in the first part of the paper admits a *log-potential*. That is, the multiproduct-firm pricing game with logged payoffs admits a potential. More generally, classic examples of games admitting an ordinal potential that is not a potential—such as the homogeneous-goods Cournot model with symmetric firms (Kukushkin, 1994; Monderer and Shapley, 1996b) and thus the lottery contest with symmetric players—also admit a log-potential.

We introduce the novel concept of a *transformed potential*: We say that a normal-form game admits a transformed potential if there exists a strictly monotone transformation function G such that the game that results from applying this transformation to all players' payoffs admits a potential. The advantage of this approach is that, for a given transformation function (such as the logarithm), Monderer and Shapley (1996b)'s cross-partial derivatives test can be applied.

In the second part of the paper (Section 3), we address the following two related questions. What classes of demand systems give rise to a transformed-potential multiproduct-firm pricing game? What is the associated set of transformation functions? In answering these questions, we require that the demand system induces a game admitting a transformed potential regardless of the ownership structure of products (i.e., which product is offered by which firm) and the vector of marginal costs. With a slight abuse of terminology, we will often say that such a demand system admits a transformed potential. Solving (systems of) ordinary and partial differential equations, we show that the only classes of demand systems admitting a transformed potential are of the "generalized linear" or IIA forms. In the latter case, the corresponding transformation function is of the log type, whereas it is of the linear type in the former case.

In the final part of the paper (Section 4), we relax the requirement that the demand system induces a game admitting a transformed potential *regardless of the ownership structure* by, instead, fixing the ownership structure of products. Although we continue to find that the only admissible transformations are of the linear and log types, we identify a richer class of demand systems. For a given ownership structure, the class of demand systems that corresponds to linear transformation functions continues to be of the generalized linear form,

[2]For example, Monderer and Shapley (1996b) write, "Unlike (weighted) potential games, ordinal potential games are not easily characterized. We do not know of any useful characterization [...] for differentiable ordinal potential games." In recent work, Ewerhart (2020) provides derivatives-based necessary conditions for a smooth game to admit an ordinal potential. However, as those conditions are not sufficient, they do not permit a complete characterization of smooth ordinal potential games. Moreover, Ewerhart's derivatives-based test must be performed at a Nash equilibrium action profile, which further limits its applicability for the questions addressed in this paper.

albeit in a slightly richer form which we completely characterize.

The system of partial differential equations that characterizes the class of demand systems corresponding to log transformation functions is hard to solve in general. We provide a complete solution for the case of two firms. In that case, the demand system has a nest structure that permits patterns of substitutability and complementarity that go beyond those implied by the IIA property. In particular, the nest structure allows products to be complements within a firm, but substitutes across firms, as would arise in models featuring "one-stop shopping" (Stahl, 1982; Bliss, 1988; Chen and Rey, 2012). Although we are not able to provide a complete solution of the general case of three or more firms, we provide two rich classes of demand systems that admit a log-potential for a given ownership structure. In the first such class, each firm owns one or more entire nests of products, so that competition takes place across nests, but not within nests. In the second such class, each firm owns products in only one nest and may face competition from rival firms in that same nest, as well as from firms in different nests.[3]

Related literature. Our paper is motivated by, and contributes to, the literature on multiproduct-firm pricing games with horizontally differentiated products.[4] As a multiproduct firm's profit function typically fails to be quasi-concave in own price, Caplin and Nalebuff (1991)'s existence result for single-product-firm pricing games does not extend. As a result, equilibrium existence had, until recently, been shown only in special cases of demand systems satisfying some variants of the IIA property: Multinomial logit demand (Spady, 1984; Konovalov and Sándor, 2010), CES demand (Konovalov and Sándor, 2010), and nested multinomial logit demand where each firm owns a nest of products (Gallego and Wang, 2014). In recent work, Nocke and Schutz (2018) adopt an aggregative games approach to unify and extend those results to the larger class of demand systems that can be derived from (multi-stage) discrete/continuous choice, under some restrictions on the relationship between the nest and ownership structures. The present paper further generalizes these earlier equilibrium existence results, and more substantially, allows products to be not only substitutes but also (local) complements, depending on the level of prices.[5]

Our paper also contributes to the literature on potential games, pioneered by Slade (1994)

[3]For models with competition within and across nests, under the assumption of nested CES or multinomial logit demand with substitutes, see Nocke and Schutz (forthcoming) and Garrido (forthcoming).

[4]The focus on horizontally differentiated products is shared by the empirical industrial organization literature (e.g., Berry, 1994; Berry, Levinsohn, and Pakes, 1995; Nevo, 2001; Berry, Levinsohn, and Pakes, 2004; Miller and Weinberg, 2017). There is a separate theoretical literature on multiproduct-firm oligopoly with pure vertical product differentiation (see Champsaur and Rochet, 1989; Johnson and Myatt, 2003, 2006).

[5]For an equilibrium existence result with complements and substitutes, see Quint (2014). His framework, however, differs from ours in two important ways: First, on the demand side, products are perfect complements within a nest and substitutes across nests; second, he restricts attention to single-product firms.

and Monderer and Shapley (1996b). Potential games have been shown to have desirable properties. For example, the Nash equilibrium that maximizes the potential function satisfies the finite improvement property (Monderer and Shapley, 1996b), the fictitious play property (Monderer and Shapley, 1996a), local asymptotic stability (Slade, 1994), and is robust to incomplete information (Ui, 2001).

Closer to our work, Slade (1994) proposes a class of inverse demand systems for differentiated products such that the induced single-product firm quantity-setting game admits a potential. She does not, however, provide a complete characterization of the demand systems satisfying that property. By contrast, we introduce the concept of a transformed potential and characterize the set of demand systems such that the induced multiproduct-firm pricing game admits a transformed potential. In unpublished work, Quint (2006) notes that the single-product-firm pricing game with logged payoffs, multinomial logit demand, and costless production admits a potential. We show that this property holds for a considerably larger class of demand systems with multiproduct firms and costly production.

Building on the seminal paper of Gentzkow (2007), there is a growing literature in empirical industrial organization focusing on the estimation of consumer demand in the presence of complementarities. Recent contributions include Thomassen, Smith, Seiler, and Schiraldi (2017), Iaria and Wang (2020), Ershov, Orr, and Laliberté (forthcoming), and Wang (2024). Unlike the existing literature on multiproduct-firm oligopoly, our approach to equilibrium existence can accommodate such complementarities.

Price-dependent patterns of substitutability/complementarity are at the heart of Rey and Tirole (2019)'s analysis of the effects of cooperative price caps. Price caps (or floors) can be shown to break the convex-valuedness of the best-response correspondence, resulting in serious issues for existing approaches to equilibrium existence based on the Kakutani fixed-point theorem or on aggregative games techniques. By contrast, our equilibrium existence results extend readily to competition in the presence of arbitrary price caps (or floors).

2 Multiproduct-Firm Oligopoly with IIA Demand

In this section, we use a potential games approach to study a multiproduct-firm pricing model where demand satisfies the IIA property, and products can be (local) complements or substitutes, depending on the level of prices. The potential games approach allows us to establish equilibrium existence under minimal assumptions.

2.1 The Model

Consider an industry with a finite set of differentiated products $\mathcal{N}$. The representative consumer's quasi-linear indirect utility is given by:

$$y + V(p) = y + \Psi\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right),$$

where y denotes income, p_j the price of product j, and Ψ and h_j are differentiable functions of a single variable. Roy's identity yields the demand for product i:

$$D_i(p) = -h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right).$$

Well-known special cases of this class of demand system include multinomial logit demand (with $\Psi(H) = \log(1 + H)$ and $h_i(p_i) = \exp[(a_i - p_i)/(\lambda)]$) and CES demand ($\Psi = \log$ and $h_i(p_i) = a_i p_i^{1-\sigma}$). Another special case, which is commonly used in the empirical industrial organization literature (see Ciliberto, Murry, and Tamer, 2021; Betancourt, Hortacsu, Oery, and Williams, 2022; Miller, Osborne, Sheu, and Sileo, 2023), is nested logit with two nests, one for the inside goods and one for the outside option ($\Psi(H) = \log(1 + H^\alpha)$ and $h_i(p_i) = \exp[(a_i - p_i)/(\lambda)]$). More generally, Nocke and Schutz (2018) provide necessary and sufficient conditions for this demand system to be derivable from multistage discrete/continuous choice. The choice process is sequential, with the consumer first observing the value of an outside option, and deciding whether to take it. If he does not take it, he observes a vector of product-specific taste shocks, and chooses the product that delivers the highest indirect utility. Finally, he decides how much of that product to consume. Under this micro-foundation, $\log h_j$ corresponds to the mean utility delivered by good j, whereas the function Ψ reflects the distribution of the value of the outside option. Nocke and Schutz (2018)'s necessary and sufficient conditions, which we assume to hold throughout, are:

(i) Each h_i is $\mathcal{C}^1$, strictly positive, strictly decreasing, and log-convex.

(ii) Ψ is $\mathcal{C}^1$ with non-negative derivative, and $H \mapsto H\Psi'(H)$ is non-decreasing.

To streamline the exposition, we strengthen condition (ii) slightly, imposing that Ψ' be everywhere strictly positive.

This demand system has the IIA property as

$$D_i(p)/D_j(p) = h_i'(p_i)/h_j'(p_j)$$

is independent of the price of any third product k. Despite the demand system being derivable from discrete/continuous choice, products can be complements. Specifically, products are (local) complements if Ψ' is locally increasing and local substitutes if Ψ' is locally decreasing. The reason why complementarities can arise is that a reduction in the price of good j reduces the probability that a consumer takes the outside option, thereby potentially increasing the *ex ante* choice probability for good $k \neq j$.

On the supply side, the set of firms, $\mathcal{F}$, is a partition of the set of products, $\mathcal{N}$. We assume that there are at least two firms. Firms produce under constant returns to scale; the vector of constant unit costs for all products is denoted $c = (c_j)_{j \in \mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$.

Setting $h_j(\infty) \equiv \lim_{p_j \to \infty} h_j(p_j)$, and adopting the convention that the sum of an empty collection of reals is equal to zero, the profit of firm f is given by:

$$\pi^f(p) = \sum_{\substack{k \in f: \\ p_k < \infty}} (p_k - c_k)(-h'_k(p_k)) \Psi' \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right), \ \forall p \in (0, \infty]^{\mathcal{N}},$$

As in Nocke and Schutz (2018), the compactification of action sets permitted by infinite prices will be useful to establish existence of equilibrium. The assumption is that an infinite price on a product results in zero profit from that product. As $(p_k - c_k) h'_k(p_k) \underset{p_k \to \infty}{\longrightarrow} 0$ by log-concavity of h_k, this assumption is consistent with what one would obtain if one were to take limits in the profit function, as long as the limiting vector of industry prices has at least one finite component.[6]

Firms compete by setting prices simultaneously. For every firm $f \in \mathcal{F}$, define

$$\mathcal{P}^f \equiv \left\{ p^f \in (0, \infty]^f : \quad \sum_{\substack{k \in f: \\ p_k < \infty}} (p_k - c_k)(-h'_k(p_k)) > 0 \right\}.$$

As price vectors outside $\mathcal{P}^f$ are strictly dominated for firm f, we redefine the action set of firm f as $\mathcal{P}^f$ in the following.

2.2 Equilibrium Existence: A Potential Games Approach

For every $p \in \prod_{g \in \mathcal{F}} \mathcal{P}^g$, define

$$W(p) \equiv \Psi' \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right) \prod_{g \in \mathcal{F}} \sum_{\substack{k \in g: \\ p_k < \infty}} (p_k - c_k)(-h'_k(p_k)). \tag{1}$$

[6]Profit functions do not necessarily have a limit as *all* prices tend to infinity. Examples of demand systems where the limit does not exist include CES and the multinomial logit without outside options. See Section II.3 in the Online Appendix to Nocke and Schutz (2018) for a detailed discussion of infinite prices.

With a slight abuse of notation, let (p^f, p^{-f}) be the vector of prices when firm f sets the price vector p^f and rivals set p^{-f}. Since, for every $f \in \mathcal{F}$,

$$W(p) = \pi^f(p) \times \underbrace{\prod_{g \neq f} \sum_{\substack{k \in g: \\ p_k < \infty}} (p_k - c_k)\left(-h_k'(p_k)\right)}_{>0, \text{ independent of } p^f},$$

we have that, for every $p = (p^f, p^{-f}) \in \prod_{g \in \mathcal{F}} \mathcal{P}^g$ and $p^{f\prime} \in \mathcal{P}^f$,

$$\pi^f(p^{f\prime}, p^{-f}) - \pi^f(p^f, p^{-f}) > 0 \Longleftrightarrow W(p^{f\prime}, p^{-f}) - W(p^f, p^{-f}) > 0.$$

The function $W(\cdot)$ is therefore an ordinal potential for the multiproduct-firm pricing game defined in the previous subsection.

As shown in Monderer and Shapley (1996b), the ordinal potential can be used to obtain a simple proof of equilibrium existence: If p^* solves the maximization problem

$$\max_{p \in \prod_{g \in \mathcal{F}} \mathcal{P}^g} W(p),$$

then for every $f \in \mathcal{F}$ and $p^f \in \mathcal{P}^f$,

$$\pi^f(p^{*f}, p^{*-f}) \geq \pi^f(p^f, p^{*-f}),$$

and so p^* is a Nash equilibrium. Equilibrium existence can thus be established by showing the existence of a global maximizer of the ordinal potential.

Applying this insight to our multiproduct-firm pricing game, we obtain equilibrium existence under minimal restrictions:

Proposition 1. *Suppose that Ψ is twice differentiable. Then, for any firm partition $\mathcal{F}$ and any marginal cost vector c, the associated ordinal potential function $W(\cdot)$ has a global maximizer, and the multiproduct-firm pricing game has a pure-strategy Nash equilibrium.*

Proof. See Appendix A. □

A substantial economic contribution relative to Nocke and Schutz (2018) consists in deriving equilibrium existence results allowing for complements. In the present framework, whether products are local substitutes or complements depends on the local behavior of Ψ', and thus on the level of prices. Such price-dependent patterns of complementarity/substitutability are at the core of Rey and Tirole (2019).

Importantly, the equilibrium existence result of Proposition 1 continues to hold even in the presence of arbitrary price caps and floors. Such price caps (and floors) may arise because of regulation or, as recently advocated by Rey and Tirole (2019), due to cooperative agreements. Suppose that for all $i \in \mathcal{N}$, there exists a price cap $\overline{p}_i \leq \infty$ and a price floor $\underline{p}_i \geq c_i$ such that p_i has to satisfy $\underline{p}_i \leq p_i \leq \overline{p}_i$. As this type of regulation breaks the convex-valuedness of best responses, standard approaches to equilibrium existence based on the Kakutani fixed point theorem or aggregative games techniques do not apply.[7] By contrast, the potential games approach still delivers equilibrium existence: As $\prod_{j\in\mathcal{N}}[\underline{c}_j, \overline{c}_j]$ is compact, the potential function continues to have a global maximizer, and so a Nash equilibrium exists.[8]

We close this section by discussing some of the more technical aspects of Proposition 1, providing first an overview of the key steps of its proof. Suppose first that products are never complements, so that price vectors that contain components at or below marginal cost are strictly dominated. We can thus restrict the domain of the potential function W to the set $\prod_{j\in\mathcal{N}}(c_j, \infty] \cap \prod_{f\in\mathcal{F}} \mathcal{P}^f$. The next step consists in showing that the function W can be extended in a continuous way to the set $\prod_{j\in\mathcal{N}}[c_j, \infty]$, and has a global maximizer p^* on that set. This p^* must necessarily be an element of $\prod_{f\in\mathcal{F}} \mathcal{P}^f$, for otherwise, $W(p^*)$ would be equal to zero, and so W could not be maximized at p^*. A difficulty in establishing existence of p^* involves showing that the extension of $W(\cdot)$ to $\prod_{j\in\mathcal{N}}[c_j, \infty]$ is continuous even when all components of the price vector are infinite—see Lemma C in the appendix for details.

To allow products to be complements involves additional technical difficulties as a firm might want to price some of its products at zero (and thus below marginal cost) to boost the demand for its other products. This is problematic as the demand system is not defined at such prices. The assumption that Ψ is twice differentiable is a weak technical condition ensuring that such a pricing incentive does not exist.

A more technical contribution of Proposition 1 relative to Nocke and Schutz (2018) consists in deriving equilibrium existence under weaker regularity and monotonicity assumptions: In the baseline of Nocke and Schutz (2018), it was assumed that Ψ is equal to the logarithm, and each h_i is $\mathcal{C}^3$ and such that the elasticity of $-h_i'$ is non-decreasing.[9] Without such regularity and monotonicity assumptions, it is easy to construct examples of multiproduct-firm pricing games with IIA demand where best responses are neither convex-valued nor monotone. Despite such classic conditions failing to hold, Proposition 1 implies that those pricing games have a Nash equilibrium.

Finally, the potential games approach provides a new method to compute equilibria.

[7] Recall from Spady (1984) and Hanson and Martin (1996) that multinomial logit profit functions can fail to be quasi-concave. In the presence of price caps or floors, this failure of quasi-concavity can result in the failure of uni-modality.

[8] More generally, an equilibrium exists provided action sets are closed.

[9] In their online appendix, Nocke and Schutz (2018), allow Ψ to differ from the logarithm. However, their existence proof requires numerous additional technical assumptions. (See Assumption (iii) in their online appendix.)

Instead of solving a multidimensional fixed point problem (as with the best-response approach) or a nested fixed point problem (as with the aggregative games approach of Nocke and Schutz, 2018), it involves finding the global maximizer of the ordinal potential function W.

3 Transformed Potentials and Demand Systems

The multiproduct-firm pricing game analyzed in the previous section has an important feature: Despite that game not having a potential, the game resulting from taking a well-chosen monotone transformation of the payoff functions does have a potential.

Specifically, the normal form game with payoff functions $\log \pi^f$ for every $f \in \mathcal{F}$ has a potential: $U \equiv \log W$, where W is the ordinal potential defined in equation (1). That is, the demand system of Section 2 has the following property: There exists a transformation function G (here, $G \equiv \log$) such that—regardless of the vector of marginal costs c and of the firm partition $\mathcal{F}$—the normal form game with payoff function $G \circ \pi^f$ for every firm f has a potential. In such a case, we say that D admits a *transformed potential* or, more specifically, a *G-potential*. In this section, we fully characterize the set of demand systems that admit a transformed potential and provide admissible transformation functions.

3.1 Demand Systems Admitting Transformed Potentials: A Complete Characterization

Let the demand system D be a continuous mapping from $\mathbb{R}^{\mathcal{N}}_{++}$ to $\mathbb{R}^{\mathcal{N}}_{+}$. Let $\mathcal{Q} \equiv \{p \in \mathbb{R}^{\mathcal{N}}_{++} : D(p) \in \mathbb{R}^{\mathcal{N}}_{++}\}$ be the set of price vectors at which the demand for all products is strictly positive. By continuity, $\mathcal{Q}$ is open.

We impose the following technical restrictions on the demand system D. The set $\mathcal{Q}$ is non-empty and convex. Moreover, D is $\mathcal{C}^2$ on $\mathcal{Q}$ and satisfies Slutsky symmetry and strict monotonicity: For all $p \in \mathcal{Q}$ and all $i, j \in \mathcal{N}$, $\partial_i D_j(p) = \partial_j D_i(p)$ and $\partial_i D_i(p) < 0$.[10] It also satisfies non-zero substitution almost everywhere: For all $i, j \in \mathcal{N}$ and almost every $p \in \mathcal{Q}$, $\partial_j D_i(p) \neq 0$. We also assume that for every product $i \in \mathcal{N}$ there exists a price vector $p \in \mathcal{Q}$ such that $\partial_i [p_i D_i(p)] < 0$; that is, the revenue on product i is not everywhere increasing in the price of that product. Slutsky symmetry and the convexity of $\mathcal{Q}$ imply the existence of a function V such that $\partial_i V(p) = -D_i(p)$ for every $p \in \mathcal{Q}$ and $i \in \mathcal{N}$. We assume that the level sets of V are connected surfaces, in the sense that any two points on the same level set

[10]Notation: $\partial_i \kappa$ denotes the partial derivative of the function κ with respect to its ith argument; $\partial^2_{ij} \kappa$ denotes the cross-partial derivative with respect to the ith and jth arguments.

can be connected by a continuously differentiable path.[11]

In the following, we seek to characterize potential functions on the set $\mathcal{Q}$.[12] We restrict attention to transformation functions G that have the following two properties: First, those functions are defined on an interval of strictly positive reals that include all attainable, strictly positive profit levels; second, those functions G are $\mathcal{C}^2$ with $G' > 0$. Given the classes of demand systems identified in the theorem below, the fact that we are not attempting to define transformation functions over non-positive reals turns out to be irrelevant.

Theorem 1. *Let D be a demand system. The following assertions are equivalent:*

(a) *D admits a transformed potential.*

(b) *At least one of the following assertions holds true:*

(i) The demand system D takes the IIA form

$$D_i(p) = -h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right). \tag{2}$$

(ii) The demand system D takes the generalized linear form

$$D_i(p) = -h_i'(p_i) + \sum_{j\neq i} \alpha_{ij} p_j, \tag{3}$$

with $\alpha_{ij} = \alpha_{ji}$ for every i, j.

If assertion (i) (resp., assertion (ii)) holds, then the logarithm (resp., the identity function) is an admissible transformation function for demand system D.

The theorem thus shows that D admits a transformed potential if and only if one (or both) of the following conditions holds. First, D takes the IIA form analyzed in Section 2. In this case, D admits a log-potential. Second, D takes the generalized linear form, a special case of which is the linear demand system of Bowley (1924) and Shubik and Levitan (1980). In this case, D admits an identity-potential.

We close this subsection by providing expressions for the associated potential functions. The potential function for part (b)-(i) of the theorem can be found by taking the logarithm

[11] This assumption will later allow us to invoke results by Goldman and Uzawa (1964) and Anderson, Erkal, and Piccinin (2020) to integrate systems of partial differential equations. If $\mathcal{Q} = \mathbb{R}^{\mathcal{N}}_{++}$, then the assumption is automatically satisfied if V is convex, i.e., if the demand system D can be derived from quasi-linear utility maximization.

[12] The differential techniques we are using in this paper do not allow us to deal with kinks in demand systems. Such kinks typically occur at price vectors at which the demand for one product vanishes.

of the ordinal potential function in equation (1):

$$U(p) = \log \Psi' \left(\sum_{j \in \mathcal{N}} h_j(p_j) \right) + \sum_{f \in \mathcal{F}} \log \left(\sum_{j \in f} (p_j - c_j)(-h_j'(p_j)) \right). \tag{4}$$

A potential function for part (b)-(ii) can be obtained by integrating the payoff gradient:

$$U(p) = \sum_{k \in \mathcal{N}} (p_k - c_k)(-h_k'(p_k)) + \frac{1}{2} \sum_{\substack{j,k \in \mathcal{N} \\ j \neq k}} \alpha_{jk} p_j p_k + \frac{1}{2} \sum_{f \in \mathcal{F}} \sum_{\substack{j,k \in \mathcal{N} \\ j \neq k}} \alpha_{jk} (p_k - c_k)^2. \tag{5}$$

3.2 Proof of the Theorem

The fact that (b) implies (a) and that the logarithm (resp., the identity function) is an admissible transformation function for IIA demand systems (resp., generalized linear demand systems) follows immediately by noting that the gradient of the potential function defined in equation (4) (resp., equation (5)) is equal to the transformed payoff gradient (see Monderer and Shapley, 1996b).

In the remainder of this subsection, we show that (a) implies (b). Suppose that the demand system D admits a G-potential. We begin by introducing new notation. For every $i \in \mathcal{N}$, let $\overline{\pi}_i \equiv \sup_{p \in \mathcal{Q}} p_i D_i(p)$ be the supremum of the revenue from product i, and let $\overline{\pi} \equiv \max_{i \in \mathcal{N}} \overline{\pi}_i$. Define

$$\pi_i : \ (p, c_i) \in \{(p, c_i) \in \mathcal{Q} \times \mathbb{R}_{++} : \ p_i > c_i\} \mapsto (p_i - c_i) D_i(p).$$

The range of π_i is the open interval $(0, \overline{\pi}_i)$.

For every $\pi \in (0, \overline{\pi}_i)$, let

$$Q_i(\pi) = \{p \in \mathcal{Q} : \ p_i D_i(p) > \pi\}.$$

For every π, $Q_i(\pi)$ is non-empty and open, and the set function $Q_i(\cdot)$ is non-increasing: $Q_i(\pi) \subseteq Q_i(\pi')$ whenever $\pi \geq \pi'$. Moreover, $p \in Q_i(\pi)$ if and only if there exists $c_i < p_i$ such that $\pi_i(p, c_i) = \pi$.

Let $\varphi(\pi) \equiv \pi G'(\pi)$ for every π. Applying Theorem 4.5 in Monderer and Shapley (1996a), we show that φ solves a certain parameterized ordinary differential equation:

Lemma 1. *For every $i, j \in \mathcal{N}$ with $i \neq j$, there exists a function $\kappa_{ij}(\cdot)$ such that for every $\pi \in (0, \overline{\pi}_i)$ and $p \in Q_i(\pi)$,*

$$\partial_j D_i \left(1 + \pi \frac{\partial_i D_i}{D_i^2} \right) \varphi'(\pi) + \left(\frac{\partial_{ij}^2 D_i}{D_i} - \frac{\partial_i D_i \partial_j D_i}{D_i^2} \right) \varphi(\pi) = \kappa_{ij}(p), \tag{6}$$

where the function D_i and its partial derivatives are all evaluated at p.

Proof. See Appendix B. □

Exploiting Lemma 1, we characterize the admissible transformation functions $G(\pi)$ for π sufficiently close to 0:

Lemma 2. *There exist constants $\hat{\pi} > 0$, A, B, and C such that $B + C\pi > 0$ and*

$$G(\pi) = A + B \log \pi + C\pi \tag{7}$$

for every $\pi \in (0, \hat{\pi})$.

Proof. See Appendix B. □

Using again Theorem 4.5 in Monderer and Shapley (1996a) and the above transformation functions, we show that the demand system must satisfy certain partial differential equations:

Lemma 3. *If $B \neq 0$ in equation (7), then for every $p \in \mathcal{Q}$,*

$$\forall (i,j,k) \in \mathcal{N}^3 \text{ with } k \neq i,j, \quad \partial_k \frac{D_i(p)}{D_j(p)} = 0,$$

$$\forall (i,j) \in \mathcal{N}^2, \quad \partial^2_{ij} \log \frac{D_i(p)}{D_j(p)} = 0.$$

If $C \neq 0$ in equation (7), then for every $i,j,k \in \mathcal{N}$ with $k \neq i,j$ and every $p \in \mathcal{Q}$, $\partial^2_{ik} D_j(p) = 0$.

Proof. See Appendix B. □

Integrating the system of partial differential equations from the second part of the previous lemma (which is straightforward) as well as from the first part (which relies on earlier results by Goldman and Uzawa (1964) and Anderson, Erkal, and Piccinin (2020)) yields:

Lemma 4. *If $B \neq 0$ in equation (7), then the demand system D takes the IIA form of equation (2) on the domain $\mathcal{Q}$.*
If $C \neq 0$ in equation (7), then the demand system D takes the generalized linear form of equation (3) on the domain $\mathcal{Q}$.

Proof. See Appendix B. □

4 Nested Demand Systems: An Exploration

In the previous section, we fully characterized the demand systems and transformation functions that give rise to a potential game *regardless of the firm partition*. In this section, we analyze whether, for a fixed firm partition $\mathcal{F}$, there are richer demand systems that induce a multiproduct-firm pricing game admitting a transformed potential.

We say that $(D, \mathcal{F})$ admits a transformed potential if there exists a transformation function G such that, for every marginal cost vector c, the multiproduct-firm pricing game with payoff function $G \circ \pi^f$ for any $f \in \mathcal{F}$ has a potential. We continue to focus on demand systems and transformation functions satisfying the technical conditions introduced in Section 3.1. We further confine attention to demand systems that are $|\mathcal{F}|$ times continuously differentiable.

Proposition 2. *Let D be a demand system and $\mathcal{F}$ a firm partition. The following assertions are equivalent:*

(a) *$(D, \mathcal{F})$ admits a transformed potential.*

(b) *At least one of the following assertions holds true:*

(i) The demand system D satisfies the following properties: For every $f, g \in \mathcal{F}$ with $f \neq g$, $i, j \in f$, and $k \in g$, $\partial_k D_i / D_j = 0$ and $\partial^2_{ik} \log D_i / D_k = 0$.

(ii) The demand system D takes the following form: For any $i \in f \in \mathcal{F}$,

$$D_i(p) = -\partial_i \psi^f(p^f) + \sum_{\substack{\mathcal{F}' \subseteq \mathcal{F}: \\ f \in \mathcal{F}'}} \sum_{\substack{\iota \in \prod_{g \in \mathcal{F}'} g: \\ \iota(f) = i}} \alpha(\iota) \prod_{\substack{g \in \mathcal{F}': \\ g \neq f}} p_{\iota(g)},$$

where p^f is the vector of prices set by firm f.

If assertion (i) (resp., assertion (ii)) holds, then the logarithm (resp., the identity function) is an admissible transformation function for demand system D.

Proof. See Appendix C. □

The class of demand systems in part (b)-(ii) of the proposition is more general than that in part (b)-(ii) of Theorem 1 in the following sense. First, D_i is now allowed to depend non-linearly not only on p_i but also on p^f, the vector of prices of the firm owning product i. Second, for $k \notin f$, the substitution effect $\partial_k D_i$ is not necessarily constant, in that it is allowed to depend on the prices set by firms that own neither product i nor product k. The associated potential function is:

$$U(p) = \sum_{f \in \mathcal{F}} \sum_{j \in f} (p_j - c_j) \left(-\partial_j \psi^f(p^f)\right) + \sum_{\mathcal{F}' \subseteq \mathcal{F}} \sum_{\iota \in \prod_{g \in \mathcal{F}'} g} \alpha(\iota) \prod_{g \in \mathcal{F}'} p_{\iota(g)}. \tag{8}$$

In contrast to part (b)-(i) of Theorem 1, part (b)-(i) of the proposition does not fully characterize the resulting demand system, but instead provides a system of partial differential equations that the demand system must solve. Unfortunately, we have not been able to solve this system in the general case with three or more firms.

We now fully characterize part (b)-(i) of Proposition 2 for the case of two firms:

Proposition 3. *Suppose $|\mathcal{F}| = 2$. Then, $(D, \mathcal{F})$ admits a log-potential if and only if the demand system D takes the following form: For every $i \in f \in \mathcal{F}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i \psi^f(p^f)\Psi'\left(\sum_{g\in\mathcal{F}} \psi^g(p^g)\right),$$

where p^g is the vector of prices set by firm g.

Proof. See Appendix D. □

The associated potential function is:

$$U(p) = \log \Psi'\left(\sum_{f\in\mathcal{F}} \psi^f(p^f)\right) + \sum_{f\in\mathcal{F}} \log\left(\sum_{j\in f}(p_j - c_j)\left(-\partial_j \psi^f(p^f)\right)\right). \tag{9}$$

The class of demand systems characterized in Proposition 3 permits more flexibility than that in part (b)-(i) of Theorem 1 in that the $\psi^g(\cdot)$ functions do not need to be additively separable in the components of the price vector p^g. A possible micro-foundation for this new demand system is a three-stage discrete/continuous choice process: First, the consumer decides whether to take up the outside option (as a function of the realization of the taste shock for that option); if not, second, the consumer chooses from which firm to purchase (as a function of the realizations of the taste shocks for the two firms); finally, the consumer chooses which products to purchase (and how much) from the selected firm (as a function of the product-level taste shocks). With this micro-foundation, $\log \psi^f$ can be interpreted as the consumer's mean utility of choosing firm f, whereas Ψ reflects the distribution of the taste shock for the outside option. A special case of the class of demand systems characterized in Proposition 3 is a nested multinomial logit (or nested CES) demand system where each firm owns one or several nests of products, and each nest can consist of several sub-nests, etc.

The fact that ψ^g need not be additively separable in p^g permits some substitution patterns that go beyond those implied by the IIA property. Specifically, the ratio of demands for goods i and j can depend on the price of a third product k provided that product k is owned by a firm that also owns at least one of the two products i and j.

The class of demand systems of Section 2 had the property that, at any given vector of prices, all products were either local substitutes or local complements to one another. The new class of demand systems of Proposition 3 permits more flexibility in this regard: For example, product 1 could be a complement to product 2 and a substitute to product 3, with all three products owned by the same firm, and at the same time a substitute to all products owned by the rival firm. Such demand patterns frequently arise through "one-stop shopping," where products offered by different stores are substitutes, but products offered by the same store can be complements.

We close this section by discussing the case of three or more firms, providing examples of rich classes of demand systems that admit a log-potential for a given firm partition f. The first example is the class of demand systems identified in Proposition 3, but with an arbitrary number of firms:

Proposition 4. *Let $\mathcal{F}$ be a firm partition. Then, $(D, \mathcal{F})$ admits a log-potential if the demand system D takes the following form: For every $i \in f \in \mathcal{F}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i \psi^f(p^f) \Psi' \left(\sum_{g \in \mathcal{F}} \psi^g(p^g) \right).$$

Proof. The result follows immediately by noticing that the gradient of the potential function in equation (9) coincides with the transformed payoff gradient. □

That first example has the feature that a product of firm f is an equally good substitute (or complement) to a product of firm f' as to one of firm f''. The second example relaxes that feature:

Proposition 5. *Let $\mathcal{F}$ be a firm partition and $\mathcal{E}$ a partition of $\mathcal{F}$. Then, $(D, \mathcal{F})$ admits a log-potential if the demand system D takes the following form: For every $i \in f \in e \in \mathcal{E}$ and $p \in \mathcal{Q}$,*

$$D_i(p) = -\partial_i \psi^f(p^f) \Psi^{e\prime} \left(\sum_{g \in e} \psi^g(p^g) \right) \Psi' \left[\sum_{\epsilon \in \mathcal{E}} \Psi^{\epsilon} \left(\sum_{g \in \epsilon} \psi^g(p^g) \right) \right].$$

Proof. The result follows immediately by defining the potential

$$U(p) = \log \Psi' \left[\sum_{\epsilon \in \mathcal{E}} \Psi^{\epsilon} \left(\sum_{g \in \epsilon} \psi^g(p^g) \right) \right] + \sum_{\epsilon \in \mathcal{E}} \log \Psi^{\epsilon\prime} \left(\sum_{g \in \epsilon} \psi^g(p^g) \right) + \sum_{\epsilon \in \mathcal{E}} \sum_{g \in \epsilon} \log \left(\sum_{j \in g} (p_j - c_j) \left(-\partial_j \psi^g(p^g) \right) \right)$$

and noticing that its gradient coincides with the transformed payoff gradient. □

One interpretation for this class of demand systems is that consumers make multi-stage discrete/continuous choices, with different subsets of firms having their products in different nests. Under this interpretation, $\log \Psi^e$ is the mean utility delivered by nest e.

5 Conclusion

In this paper, we have made several contributions. First, we have proven existence of equilibrium for multiproduct-firm pricing games with IIA demand under minimal restrictions on demand. Importantly, the demand framework has the property that, depending on the level of prices, products can be local substitutes or complements. Moreover, the equilibrium existence result holds even in the presence of price caps and floors—instances in which other approaches to equilibrium existence face serious difficulties.

Second, we have introduced the novel concept of a transformed potential. The advantage of this new concept is that, in contrast to an ordinal potential, a cross-partial derivatives test is available for transformed potentials. We have fully characterized the class of demand systems admitting such a transformed potential regardless of the ownership structure, along with the associated transformation functions. Those demand systems are of the generalized linear or IIA types.

Third, for a given ownership structure, we have shown that the only admissible transformation functions are either of the linear or the logarithmic type. We have completely characterized the class of demand systems admitting a potential with a linear transformation function, as well as partially characterized the demand systems admitting a potential with a logarithmic transformation function. The latter demand systems can have nest structures, permitting patterns of substitutability and complementarity that go beyond those implied by the IIA property. For instance, products can be complements within a firm but substitutes across firms, as in models of one-stop shopping.

A Proof of Proposition 1

Preliminaries. We begin by introducing new notation. The function Ψ is defined over the interval $(\underline{H}, \overline{H})$, where

$$\underline{H} \equiv \sum_{j \in \mathcal{N}} h_j(\infty) \quad \text{and} \quad \overline{H} \equiv \sum_{j \in \mathcal{N}} h_j(0).$$

For every $f \in \mathcal{F}$, $p^f \in (0, \infty]^f$, and $p \in (0, \infty]^{\mathcal{N}}$, we let

$$u^f(p^f) = \sum_{\substack{k \in f \\ p_k < \infty}} (p_k - c_k)(-h_k'(p_k)),$$
$$H(p) = \sum_{j \in \mathcal{N}} h_j(p_j),$$

This allows us to rewrite firm f's profit and the ordinal potential function as follows:

$$\pi^f(p) = \Psi'(H(p)) u^f(p^f),$$
$$W(p) = \Psi'(H(p)) \prod_{g \in \mathcal{F}} u^g(p^g).$$

The following lemma will be useful to determine the limits of u^f and W as some (or all) of the prices tend to infinity:

Lemma A. *For every $j \in \mathcal{N}$ and $\alpha \in [0, 1)$,*

$$\lim_{p_j \to \infty} p_j \frac{h_j'(p_j)}{h_j(p_j)^\alpha} = 0.$$

Proof. We drop the product subscript to ease notation. Let $\alpha \in [0, 1)$ and $\phi(p) \equiv h(p)^{1-\alpha}$ for every $p > 0$. As $\phi'(p)/(1 - \alpha) = h'(p)/h(p)^\alpha$, all we need to do is show that $p\phi'(p) \underset{p \to \infty}{\longrightarrow} 0$.

As h is 0-convex (i.e., log-convex), it is ρ-convex for every $\rho \geq 0$.[13] It follows in particular that ϕ is convex. Moreover, since h is positive and decreasing, so is ϕ. This implies that $\phi(\infty) \equiv \lim_{p \to \infty} \phi(p)$ exists and is finite.

By the fundamental theorem of calculus, we have:

$$\phi(p) - \phi\left(\frac{p}{2}\right) = \int_{p/2}^{p} \phi'(t) dt \leq \int_{p/2}^{p} \phi'(p) dt = \frac{1}{2} p\phi'(p) \leq 0,$$

where the first inequality follows by the convexity of ϕ. Since $\phi(\infty)$ is finite, we have that $\phi(p) - \phi(p/2) \underset{p \to \infty}{\longrightarrow} 0$, which implies that $p\phi'(p) \underset{p \to \infty}{\longrightarrow} 0$ by the sandwich theorem. □

[13] Recall that a real-valued, strictly positive function g defined over a convex domain is said to be ρ-convex with $\rho > 0$ if g^ρ is convex, and 0 convex if $\log g$ is convex.

For what follows, it is useful to extend the domains of the functions $u^f(\cdot)$, $H(\cdot)$, and $W(\cdot)$ to price vectors for which some of the prices are equal to zero and/or some of the firms make strictly negative profits. For every firm f, let

$$\mathcal{P}_0^f \equiv \left\{ p^f \in [0,\infty]^f : \ p_j^f > 0 \text{ for every } j \text{ such that } h_j'(0) = -\infty \right\}.$$

We begin by extending the domains of u^f and H:

Lemma B. *For every $f \in \mathcal{F}$, u^f has a continuous and real-valued extension to $\mathcal{P}_0^f$. Moreover, H has a continuous and real-valued extension to $\mathcal{P}_0 \equiv \prod_{g\in\mathcal{F}} \mathcal{P}_0^g$.*

Proof. Since u^f is additively separable in p^f, all we need to do is show that, for every $j \in f$, (i) $\lim_{p_j\to\infty}(p_j - c_j)h_j'(p_j) = 0$, and (ii) if $h_j'(0) > -\infty$, then $\lim_{p_j\to 0}(p_j - c_j)h_j'(p_j)$ is finite. The former follows by Lemma A, whereas the latter holds trivially.

Next, we turn our attention to H. For every j such that $h_j'(0) > -\infty$, we have that $h_j(0) < \infty$. Hence, any such h_j can be extended in a continuous and real-valued way to $[0,\infty]$. Since H is additively separable in p, the result follows. □

Let us now extend the domain of W to $(0,\infty]^{\mathcal{N}}$ by defining for every p in that set

$$W(p) = \begin{cases} \Psi'(H(p)) \prod_{g\in\mathcal{F}} u^g(p^g) & \text{if } p_j \neq \infty \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, W is continuous on $(0,\infty)^{\mathcal{N}}$. The following lemma states that W is also continuous at price vectors containing infinite components, and extends its domain to $\mathcal{P}_0^* \equiv \mathcal{P}_0 \setminus \{0\}$:

Lemma C. *The function W has a continuous and real-valued extension to $\mathcal{P}_0^*$.*

Proof. Let $\hat{p} \in \mathcal{P}_0^*$. Suppose that $\hat{p}_k = 0$ for some k. By Lemma B, H is continuous at $\hat{p}$ and u^g is continuous at $\hat{p}^g$ for every firm g. We now show that $\lim_{H\to H(\hat{p})} \Psi'(H)$ exists and is finite. Since $\hat{p} \in \mathcal{P}_0^*$, we have that $\hat{p}_i > 0$ for some i. Moreover, since $\hat{p}_k = 0$, it follows that $H(\hat{p}) \in (\underline{H}, \overline{H})$. Therefore, $\lim_{H\to H(\hat{p})} \Psi'(H) = \Psi'(H(\hat{p}))$, which is indeed finite. Hence,

$$\lim_{p\to\hat{p}} W(p) = \Psi'(H(\hat{p})) \prod_{g\in\mathcal{F}} h^g(\hat{p}^g) \equiv W(\hat{p}).$$

Likewise, for every $\tilde{p} \in \mathcal{P}_0^*$ containing at least one finite component,

$$\lim_{p\to\tilde{p}} W(p) = \Psi'(H(\tilde{p})) \prod_{g\in\mathcal{F}} h^g(\tilde{p}^g) = W(\tilde{p}).$$

Hence, W is continuous on $\mathcal{P}_0^* \setminus \{(\infty,\infty,\dots,\infty)\}$.

Finally, we show that W is continuous at $(\infty,\infty,\dots,\infty)$. Let $(p(n))_{n\geq 0}$ be a sequence such that $p(n) \neq (\infty,\infty,\dots,\infty)$ for every n and $\lim_{n\to\infty} p(n) = (\infty,\infty,\dots,\infty)$. Then,

$H(p(n)) \underset{n\to\infty}{\longrightarrow} \underline{H}$. Since $H\Psi'(H)$ is positive and non-decreasing, it has a finite limit as H tends to $\underline{H}$. We have:

$$
\begin{aligned}
|W(p(n))| = \ & |H(p(n))\Psi'(H(p(n)))| \times \prod_{g\in\mathcal{F}} \left| \sum_{\substack{k\in g\\ p_k(n)<\infty}} \frac{(p_k(n)-c_k)(-h_k'(p_k(n)))}{H(p(n))^{\frac{1}{|\mathcal{F}|}}} \right| \\
&\leq H(p(n))\Psi'(H(p(n))) \times \prod_{g\in\mathcal{F}} \sum_{\substack{k\in g\\ p_k(n)<\infty}} \frac{|p_k(n)-c_k|\,(-h_k'(p_k(n)))}{H(p(n))^{\frac{1}{|\mathcal{F}|}}} \\
&\leq H(p(n))\Psi'(H(p(n))) \times \prod_{g\in\mathcal{F}} \sum_{\substack{k\in g\\ p_k(n)<\infty}} \frac{-p_k(n)h_k'(p_k(n))}{h_k(p_k(n))^{\frac{1}{|\mathcal{F}|}}} \\
&\underset{n\to\infty}{\longrightarrow} 0 = W(\infty,\dots,\infty),
\end{aligned}
$$

where we have used Lemma A and the fact that $\lim_{H\to\underline{H}} H\Psi'(H) \in [0,\infty)$. □

Proof of the proposition.

Proof. Let $(p(n))_{n\geq 0}$ be a sequence over $\mathcal{P}$ such that

$$\lim_{n\to\infty} W(p(n)) = \sup_{p\in\mathcal{P}} W(p).$$

For every $i\in\mathcal{N}$, the sequence $(p_i(n))_{n\geq 0}$ is either bounded or unbounded. In the former case, we can extract a subsequence that converges to some $p_i^*\in[0,\infty)$. In the latter case, we can extract a subsequence that converges to $p_i^*=\infty$. Doing so (sequentially) for every $i\in\mathcal{N}$, we obtain a subsequence $(p'(n))_{n\geq 0}$ that tends to some limiting price vector $p^*\in[0,\infty]^{\mathcal{N}}$ as n tends to infinity. To ease notation, we relabel $(p'(n))_{n\geq 0}$ as $(p(n))_{n\geq 0}$. Our goal is to show that $p^*\in\mathcal{P}$. The result will then follow by the continuity of W on $\mathcal{P}$ (see Lemma C).

We begin by showing that $p^*\in\mathcal{P}_0^*$. Clearly, $p^*\neq 0$. To see this, note that if p^* had all of its components equal to zero, then we would have $p_j(n)<c_j$ for every j for n sufficiently high, and so $p(n)$ could not belong to $\mathcal{P}$.

Assume for a contradiction that $p_i^*=0$ for a product i for which $h_i'(0)=-\infty$, and let f be the firm that owns product i. Then, $(p_i(n)-c_i)(-h_i'(p_i(n))) \underset{n\to\infty}{\longrightarrow} -\infty$. By Lemma A, $\lim_{p_j\to\infty} p_j h_j'(p_j)=0$ for every j. Hence, there exists $P_j>c_j$ such that $(p_j-c_j)(-h_j'(p_j))\leq 1$ for every $p_j\geq P_j$. Moreover, since $p_j\mapsto(p_j-c_j)(-h_j'(p_j))$ is continuous on the compact set $[c_j,P_j]$, it is bounded above by some real K_j. As $(p_j-c_j)(-h_j'(p_j))<0$ for $p_j<c_j$, this implies that $(p_j-c_j)(-h_j'(p_j))\leq\max(1,K_j)\equiv K_j'$ for every $p_j\in\mathbb{R}_{++}$. Hence, for every $n\geq 0$,

$$u^f(p^f(n)) \leq \sum_{\substack{j\in f\\ j\neq i}} K_j' + (p_i(n)-c_i)\left(-h_i'(p_i(n))\right) \underset{n\to\infty}{\longrightarrow} -\infty.$$

Thus, $u^f(p^f(n)) < 0$ for n sufficiently high, and so $p^f(n) \notin \mathcal{P}^f$, a contradiction.

Summing up, $p^* \neq 0$ and $p_j^* > 0$ for every j such that $h_j'(0) = -\infty$. It follows that $p^* \in \mathcal{P}_0^*$. Since W is continuous on $\mathcal{P}_0^*$ by Lemma C, it follows that $\sup_{p \in \mathcal{P}} W(p) = W(p^*)$.

Next, we show that $\Psi'(H(p^*))$ and $u^g(p^{g*})$ are finite and strictly positive for every firm g. Clearly, $W(p) > 0$ for every $p \in \prod_{j \in \mathcal{N}} (c_j, \infty)$, and so $W(p^*) > 0$. As, by definition, $W(p)$ is equal to zero when all the components of p are infinite, this implies that p^* has at least one finite component, so that $H(p^*) > \underline{H}$. Moreover,

$$H(p^*) = \sum_{\substack{j \in \mathcal{N} \\ p_j^* = 0}} h_j(0) + \sum_{\substack{j \in \mathcal{N} \\ p_j^* > 0}} h_j(p_j^*) < \sum_{j \in \mathcal{N}} h_j(0) = \overline{H}$$

since $p_i^* > 0$ for at least one product i and $h_j(0) < \infty$ for every product j such that $p_j^* = 0$. Hence, $H(p^*) \in (\underline{H}, \overline{H})$, and so $\Psi'(H(p^*))$ is finite and strictly positive. We therefore have:

$$W(p^*) = \underbrace{\Psi'(H(p^*))}_{\text{finite, }>0} \times \prod_{g \in \mathcal{F}} u^g(p^{g*}) > 0. \tag{10}$$

Since $p^* \in \mathcal{P}_0^*$, $u^g(p^{g*})$ is finite for every firm g. Moreover, since $p^g(n) \in \mathcal{P}^g$ for every n, p^{g*} belongs to the closure of $\mathcal{P}^g$, and so $u^g(p^{g*}) \geq 0$. Combining this with inequality (10), we conclude that $u^g(p^{g*})$ is finite and strictly positive for every g.

Finally, we show that $p_j^* > 0$ for every j. Assume for a contradiction that $p_i^* = 0$ for some product i, and let f be the firm owning that product. Since $u^f(p^{f*}) > 0$, there exists another product k owned by firm f such that $p_k^* \in (c_k, \infty)$. Since u^f is continuous, there exists $\underline{p}_k \in (c_k, p_k^*)$ such that for every $p_k \in (\underline{p}_k, p_k^*]$, $u^f(p_k, p_{-k}^{f*}) > 0$, where (p_k, p_{-k}^{f*}) denotes the vector obtained by replacing the kth element of p^{f*} by p_k. We now show that

$$W(p_k, p_{-k}^*) \leq W(p^*) \quad \forall p_k \in (\underline{p}_k, p_k^*). \tag{11}$$

To see this, define the sequence $\tilde{p}(n)$ as follows: For every $n \geq 0$ and $j \in \mathcal{N}$,

$$\tilde{p}_j(n) = \begin{cases} p_k & \text{if } j = k, \\ p_j(n) & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{p}(n) \underset{n \to \infty}{\longrightarrow} (p_k, p_{-k}^*)$. Moreover, by continuity of u^f, we have that $u^f(\tilde{p}^f(n)) > 0$ for n high enough. Since $\tilde{p}_j(n) > 0$ for every j and n, this implies that $\tilde{p}(n) \in \mathcal{P}$ for n high enough. Hence, for sufficiently high n, we have $W(\tilde{p}(n)) \leq W(p^*)$. Taking limits and using the continuity of W, we obtain condition (11).

Since $u^g(p^{g*}) > 0$ for every g, condition (11) can be rewritten as

$$\frac{\Psi'\left[H(p^*)\right] u^f(p^{f*}) - \Psi'\left[H(p_k, p_{-k}^*)\right] u^f\left(p_k, p_{-k}^{f*}\right)}{p_k^* - p_k} \geq 0.$$

Let $(p_k^n)_{n\geq 0}$ be a sequence over $(\underline{p}_k, p_k^*)$ such that $p_k^n \underset{n\to\infty}{\longrightarrow} p_k^*$. Then, for every $n \geq 0$,

$$\Psi'\left[H(p^*)\right] \frac{u^f(p^{f*}) - u^f\left(p_k^n, p_{-k}^{f*}\right)}{p_k^* - p_k^n} + u^f\left(p_k^n, p_{-k}^{f*}\right) \frac{\Psi'\left[H(p^*)\right] - \Psi'\left[H(p_k^n, p_{-k}^*)\right]}{p_k^* - p_k^n} \geq 0. \quad (12)$$

As n tends to infinity, the second term on the left-hand side tends to

$$u^f(p^{f*}) h_k'(p_k^*) \Psi''(H(p^*)),$$

where we have used the fact that h_k and Ψ' are differentiable and u^f is continuous.

As for the first term on the left-hand side, note that

$$\begin{aligned}
\frac{u^f(p^{f*}) - u^f\left(p_k^n, p_{-k}^{f*}\right)}{p_k^* - p_k^n} &= \frac{(p_k^* - c_k)(-h_k'(p_k^*)) - (p_k^n - c_k)(-h_k'(p_k^n))}{p_k^* - p_k^n} \\
&= -h_k'(p_k^*) + (p_k^n - c_k) \underbrace{\frac{h_k'(p_k^n) - h_k'(p_k^*)}{p_k^* - p_k^n}}_{\equiv \delta_k^n}.
\end{aligned}$$

Since h_k is convex and $p_k^n < p_k^*$, we have that $\delta_k^n \leq 0$ for every n. If $(\delta_k^n)_{n\geq 0}$ were unbounded, then we could extract a subsequence that diverges to $-\infty$. Along that subsequence, the left-hand side of condition (12) would then diverge to $-\infty$, which cannot be. It follows that $(\delta_k^n)_{n\geq 0}$ is bounded, and we can extract a subsequence that converges to some $\delta_k \in (-\infty, 0]$.

Assume for a contradiction that $\delta_k = 0$. By log-convexity of h_k, we have that, for every n,

$$\begin{aligned}
0 &\leq \frac{1}{p_k^* - p_k^n} \left(\frac{h_k'(p_k^*)}{h_k(p_k^*)} - \frac{h_k'(p_k^n)}{h_k(p_k^n)} \right) \\
&= \frac{h_k'(p_k^*) - h_k'(p_k^n)}{p_k^* - p_k^n} \frac{1}{h_k(p_k^*)} + h_k'(p_k^n) \frac{1}{p_k^* - p_k^n} \left(\frac{1}{h_k(p_k^*)} - \frac{1}{h_k(p_k^n)} \right), \\
&\underset{n\to\infty}{\longrightarrow} -\frac{h_k'(p_k^*)^2}{h_k(p_k^*)^2} < 0,
\end{aligned}$$

where we have taken the limit along the aforementioned subsequence and used the fact that $\delta_k = 0$. We have thus obtained a contradiction, which implies that $\delta_k < 0$.

Taking limits along the convergent subsequence in condition (12), we obtain:

$$(p_k^* - c_k) \frac{-\delta_k}{-h_k'(p_k^*)} \leq 1 - \frac{\Psi''(H(p^*))}{\Psi'(H(p^*))} u^f(p^{f*}),$$

which, since $p_k^* > c_k$ and $\delta_k < 0$, implies that

$$1 - \frac{\Psi''(H(p^*))}{\Psi'(H(p^*))} u^f(p^{f*}) > 0.$$

We now perform the same exercise for product i (for which $p_i^* = 0$). The argument used above implies that

$$\frac{\Psi'\left[H(p_i, p_{-i}^*)\right] u^f\left(p_i, p_{-i}^{f*}\right) - \Psi'\left[H(p^*)\right] u^f(p^{f*})}{p_i} \leq 0$$

for every $p_i > 0$. Let $(p_i^n)_{n\geq 0}$ be a strictly positive sequence of prices converging to zero. Using the above inequality, we obtain that, for every n,

$$\Psi'\left[H(p^*)\right] \frac{u^f\left(p_i^n, p_{-i}^{f*}\right) - u^f(p^{f*})}{p_i^n} + u^f\left(p_i^n, p_{-i}^{f*}\right) \frac{\Psi'\left[H(p_i^n, p_{-i}^*)\right] - \Psi'\left[H(p^*)\right]}{p_i^n} \leq 0. \quad (13)$$

As before, the second term on the left-hand side tends to

$$u^f(p^*) h_i'(0) \Psi''(H(p^*))$$

as n tends to infinity. Moreover,

$$\frac{u^f\left(p_i^n, p_{-i}^{f*}\right) - u^f(p^{f*})}{p_i^n} = -h_i'(0) + (p_i^n - c_i) \underbrace{\frac{h_i'(0) - h_i'(p_i^n)}{p_i^n}}_{\equiv \delta_i^n}.$$

The sequence $(\delta_i^n)_{n\geq 0}$ is non-positive. If it were unbounded, we could extract a subsequence that diverges to $-\infty$. Since $p_i^n - c_i \underset{n\to\infty}{\longrightarrow} -c_i < 0$, the left-hand side of condition (13) would then diverge to $+\infty$, which cannot be. Hence, $(\delta_i^n)_{n\geq 0}$ is bounded and we can extract a subsequence that converges to some $\delta_i \in (-\infty, 0]$.

Taking limits along the convergent subsequence in condition (13), we obtain:

$$c_i \frac{\delta_i}{-h_i'(0)} \geq 1 - \frac{\Psi''(H(p^*))}{\Psi'(H(p^*))} u^f(p^{f*}).$$

This is a contradiction since the left-hand side of the above inequality is non-positive, whereas the right-hand side is strictly positive, as shown above. Therefore, $p_j^* > 0$ for every j, and, since $u^f(p^{f*}) > 0$ for every f, we have that $p^* \in \mathcal{P}$. This concludes the proof. □

B Proof of Theorem 1

B.1 Proof of Lemma 1

Proof. Let $i, j \in \mathcal{N}$ with $i \neq j$ and $p \in \mathcal{Q}$. For any vector of marginal costs $c = (c_k)_{k\in\mathcal{N}}$ such that $c_k < p_k$ for every k, Theorem 4.5 in Monderer and Shapley (1996b), applied to the pricing game in which all firms are single-product firms and the marginal cost vector is $(c_k)_{k\in\mathcal{N}}$, implies that

$$\frac{\partial^2}{\partial p_i \partial p_j} G(\pi_i(p, c_i)) = \frac{\partial^2}{\partial p_i \partial p_j} G(\pi_j(p, c_j)). \tag{14}$$

As the right-hand side does not depend on c_{-j} while the left-hand side does not depend on c_{-i}, there exists a function $\kappa_{ij}(p)$, which is independent of the marginal cost vector, such that

$$\frac{\partial^2}{\partial p_i \partial p_j} G(\pi_i(p, c_i)) = \kappa_{ij}(p)$$

for every $c_i < p_i$.

Next, let $\pi \in (0, \overline{\pi}_i)$, and $p \in Q_i(\pi)$, and $c_i > 0$ such that $\pi_i(p, c_i) = \pi$. We have:

$$\begin{aligned}
\kappa_{ij}(p) &= \frac{\partial}{\partial p_i} \left[(p_i - c_i)\partial_j D_i G'(\pi_i(p, c_i))\right] \\
&= (p_i - c_i)\partial_j D_i \left[D_i + (p_i - c_i)\partial_i D_i\right] G''(\pi_i(p, c_i)) + \left[\partial_j D_i + (p_i - c_i)\partial^2_{ij} D_i\right] G'(\pi_i(p, c_i)) \\
&= \pi \frac{\partial_j D_i}{D_i} \left[D_i + \pi \frac{\partial_i D_i}{D_i}\right] G''(\pi) + \left[\partial_j D_i + \pi \frac{\partial^2_{ij} D_i}{D_i}\right] G'(\pi) \\
&= \partial_j D_i \left[1 + \pi \frac{\partial_i D_i}{D_i^2}\right] (\pi G''(\pi) + G'(\pi)) + \left[\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_j D_i \partial_i D_i}{D_i^2}\right] \pi G'(\pi) \\
&= \partial_j D_i \left[1 + \pi \frac{\partial_i D_i}{D_i^2}\right] \varphi'(\pi) + \left[\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_j D_i \partial_i D_i}{D_i^2}\right] \varphi(\pi). \qquad \square
\end{aligned}$$

B.2 Proof of Lemma 2

To prove Lemma 2, we split it into a series of technical lemmas. We introduce new notation. Let $\hat{p} \in \mathcal{Q}$ such that, at $p = \hat{p}$, $\partial_i(p_i D_i) < 0$ and $\partial_j D_i \neq 0$ for some $j \neq i$. (Such a price vector exists, as D is $\mathcal{C}^1$, $\partial_j D_i \neq 0$ almost everywhere, and $\partial_i(p_i D_i(p)) < 0$ for some p.) There exists $\hat{c}_i \in (0, \hat{p}_i)$ such that $(\hat{p}_i - \hat{c}_i)\partial_i D_i(\hat{p}) + D_i(\hat{p}) = 0$. Define $\hat{\pi} \equiv (\hat{p}_i - \hat{c}_i)D(\hat{p})$, and note that

$$D_i(\hat{p}) + \hat{\pi} \frac{\partial_i D_i(\hat{p})}{D_i(\hat{p})} = 0,$$

i.e., $\hat{\pi} = -D_i(\hat{p})^2 / \partial_i D_i(\hat{p})$.

We begin by solving differential equation (6) on $(0, \hat{\pi})$ for $p = \hat{p}$:

Lemma D. *There exist constants $q \in \mathbb{R}$, $s \geq 0$, and $t \in \mathbb{R}$, such that the function φ takes the form*

$$\varphi(\pi) = q\left(\hat{\pi} - \pi\right)^s + t \tag{15}$$

on the interval $(0, \hat{\pi})$.

Proof. Define

$$\alpha \equiv \partial_j D_i,\ \beta \equiv \frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_i D_i \partial_j D_i}{D_i^2},\ \text{and } \kappa \equiv \kappa_{ij}(\hat{p}),$$

where the function D_i and its derivatives are evaluated at $\hat{p}$. For every $\pi \in (0, \hat{\pi})$, we have that $\hat{p} \in Q_i(\pi)$, so that Lemma 1 applies to profit level π at price vector $\hat{p}$. Making use of the above notation, equation (6) can be rewritten as:

$$\alpha\left(1 - \frac{\pi}{\hat{\pi}}\right)\varphi'(\pi) + \beta\varphi(\pi) = \kappa.$$

Dividing both sides by $\alpha(1 - \pi/\hat{\pi})$ yields

$$\varphi'(\pi) + \frac{\beta\hat{\pi}}{\alpha(\hat{\pi} - \pi)}\varphi(\pi) = \frac{\kappa\hat{\pi}}{\alpha(\hat{\pi} - \pi)}.$$

This first-order, inhomogeneous, linear differential equation can be solved using standard techniques.

Suppose first that $\beta \neq 0$. The solutions to the corresponding *homogeneous* differential equation take the form

$$\tilde{\varphi}(\pi) = K(\hat{\pi} - \pi)^{\frac{\beta\hat{\pi}}{\alpha}},$$

where K is a constant of integration. A particular solution to the inhomogeneous differential equation is $\tilde{\varphi}(\pi) = \kappa/\beta$. Hence, φ, as a solution to the inhomogeneous differential equation, must take the form

$$\varphi(\pi) = K(\hat{\pi} - \pi)^{\frac{\beta\hat{\pi}}{\alpha}} + \frac{\kappa}{\beta}.$$

If $K = 0$, then $\varphi(\pi) = \kappa/\beta$ for every $\pi \in (0, \hat{\pi})$, and we obtain functional form (15) by setting $q = 0$, $s = 1$, and $t = \kappa/\beta$. If instead $K \neq 0$, then we obtain functional form (15) by setting $q = K$, $s = \beta\hat{\pi}/\alpha$, and $t = \kappa/\beta$. In the latter case, if s where strictly negative, then φ would tend to $\pm\infty$, implying that G would fail to be continuously differentiable at $\hat{\pi}$, a contradiction. Hence, $s \geq 0$.

Suppose instead that $\beta = 0$. Integrating $\varphi' = \kappa\hat{\pi}/(\alpha(\hat{\pi} - \pi))$ yields

$$\varphi(\pi) = -\frac{\kappa\hat{\pi}}{\alpha}\log(\hat{\pi} - \pi) + K.$$

If $\kappa \neq 0$, then we obtain the contradiction that $\varphi(\pi) \underset{\pi \to \hat{\pi}}{\longrightarrow} \pm\infty$. Hence, $\varphi(\pi) = K$ for every $\pi \in (0, \hat{\pi})$, and it is thus as in equation (15) with $t = K$, $q = 0$, and $s = 1$. $\square$

Next, we use the fact that equation (6) must hold for any p to show that φ must be affine in π for $\pi \in (0, \hat{\pi})$:

Lemma E. *There exist constants B and C such that $\varphi(\pi) = B + C\pi$ for every $\pi \in (0, \hat{\pi})$.*

Proof. By Lemma D, φ must take the form of equation (15) with $s \geq 0$ on the interval $(0, \hat{\pi})$. Assume for a contradiction that $q \neq 0$ and $s \neq 1$. Note that φ is $\mathcal{C}^2$ on $(0, \hat{\pi})$, and satisfies

$$\frac{\varphi''(\pi)}{\varphi'(\pi)} = \frac{1-s}{\hat{\pi} - \pi}.$$

Let $\tilde{\pi} \in (0, \hat{\pi})$. As $\hat{p} \in Q_i(\tilde{\pi})$, $\partial_j D_i(\hat{p}) \neq 0$, and the demand system is $\mathcal{C}^1$, there exist an open and convex set $O \subseteq \mathcal{Q}$ and an $\eta > 0$ such that $p \in Q_i(\pi)$ and $\partial_j D_i(p) \neq 0$ for every $p \in O$ and $\pi \in (\tilde{\pi} - \eta, \tilde{\pi} + \eta)$. By Lemma 1, equation (6) must hold for every such p and π. We can therefore differentiate that equation with respect to π to obtain

$$\partial_j D_i(p) \left[1 + \pi \frac{\partial_i D_i(p)}{D_i(p)^2}\right] \varphi''(\pi) + \frac{\partial^2_{ij} D_i(p)}{D_i(p)} \varphi'(\pi) = 0$$

for every $p \in O$ and $\pi \in (\tilde{\pi} - \eta, \tilde{\pi} + \eta)$. Dividing both sides by $\varphi'(\pi)$ and using the above expression for φ''/φ' yields

$$\partial_j D_i(p) \frac{1-s}{\hat{\pi}} \frac{1 + \pi \frac{\partial_i D_i(p)}{D_i(p)^2}}{1 - \frac{\pi}{\hat{\pi}}} + \frac{\partial^2_{ij} D_i(p)}{D_i(p)} = 0.$$

As the above condition must hold for every $p \in O$ and $\pi \in (\tilde{\pi} - \eta, \tilde{\pi} + \eta)$ and $(1-s)\partial_j D_i(p) \neq 0$, it follows that

$$\frac{\partial_i D_i(p)}{D_i(p)^2} = -\frac{1}{\hat{\pi}} \tag{16}$$

$$\text{and } \partial^2_{ij} D_i(p) = -\frac{1-s}{\hat{\pi}} D_i(p) \partial_j D_i(p) \tag{17}$$

for every $p \in O$.

Condition (16) can be rewritten as $\partial_i(1/D_i(p)) = 1/\hat{\pi}$. As it holds for every p in the open and convex set O, there exists a $\mathcal{C}^2$ function ϕ such that

$$\frac{1}{D_i(p)} = \frac{p_i}{\hat{\pi}} + \phi(p_{-i})$$

for every $p \in O$. Differentiating this with respect to p_j yields

$$-\frac{\partial_j D_i(p)}{D_i(p)^2} = \partial_j \phi(p_{-i}),$$

i.e., $\partial_j D_i(p) = -D_i(p)^2 \partial_j \phi(p_{-i})$. Further differentiating with respect to p_i, we obtain:

$$\partial^2_{ij} D_i(p) = -2 D_i(p) \partial_i D_i(p) \partial_j \phi(p_{-i}) = \frac{2}{\hat{\pi}} D_i^3(p) \partial_j \phi(p_{-i}),$$

where we have used equation (16) to obtain the second equality. Hence,

$$\frac{\partial^2_{ij} D_i(p)}{\partial_j D_i(p)} = -\frac{2}{\hat{\pi}} D_i(p)$$

for every $p \in O$. Combining this with condition (17), we obtain that $1 - s = 2$, i.e., $s = -1$, which is a contradiction, as s must be non-negative. □

We are now in a position to prove Lemma 2. By Lemma E, we have that $G'(\pi) = C + B/\pi$ for every $\pi \in (0, \hat{\pi})$. Hence, for some constant of integration A, $G(\pi) = A + B \log \pi + C\pi$. Moreover, as G' must be strictly positive, it must be that $B + C\pi > 0$ for every $\pi \in (0, \hat{\pi})$.

B.3 Proof of Lemma 3

Proof. Let $p \in \mathcal{Q}$, $i, j \in \mathcal{N}$, and $k, l \in \mathcal{N} \setminus \{i, j\}$, where i may or may not be equal to j, and k may or may not be equal to l. Let $f = \{i, j\}$ and $g = \{k, l\}$, and consider the firm partition $\mathcal{F} \equiv \{f, g, \mathcal{N} \setminus (f \cup g)\}$. For every $i' \in \mathcal{N} \setminus (f \cup g)$, fix some $c_{i'} \in (0, p_{i'})$. Choose $c_j \in (0, p_j)$ such that $\pi_j(p, c_j) < \hat{\pi}$ and, if $i \neq j$, let $c_i = p_i$. Similarly, choose $c_l \in (0, p_l)$ such that $\pi_l(p, c_l) < \hat{\pi}$ and, if $k \neq l$, let $c_k = p_k$. We have thus defined a multiproduct-firm pricing game. Note that, by construction, $\pi^f(p) \in (0, \hat{\pi})$ and $\pi^g(p) \in (0, \hat{\pi})$, where $\pi^h(p) \equiv \sum_{n \in f} (p_n - c_n) D_n(p)$ for $h \in \{f, g\}$. That is, both firms' profits are within the domain to which Lemma 2 applies. Moreover, the firms' profits remain in that domain for small perturbations of the marginal cost vector.

By Theorem 4.5 in Monderer and Shapley (1996b),[14] we have that

$$\frac{\partial^2}{\partial p_i \partial p_k} G\left[\pi^f(p)\right] = \frac{\partial^2}{\partial p_i \partial p_k} G\left[\pi^g(p)\right].$$

If $i \neq j$, then, by Lemma 2,

$$\begin{aligned}
\partial^2_{ik} G(\pi^f) &= \partial^2_{ik}\left[B \log \pi^f + C\pi^f\right] \\
&= B\partial_k \frac{D_i + (p_j - c_j)\partial_i D_j}{(p_j - c_j) D_j} + C\partial_k\left[D_i + (p_j - c_j)\partial_i D_j\right] \\
&= B\left[\frac{1}{p_j - c_j}\partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j\right] + C\left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j\right].
\end{aligned}$$

If instead $i = j$, then we obtain the same expression using again Lemma 2:

$$\begin{aligned}
\partial^2_{ik} G(\pi^f) &= B\partial^2_{ik} \log D_j + C\left[\partial_k D_j + (p_j - c_j)\partial^2_{ik} D_j\right] \\
&= B\left[\frac{1}{p_j - c_j}\partial_k \frac{D_i}{D_j} + \partial^2_{ik} \log D_j\right] + C\left[\partial_k D_i + (p_j - c_j)\partial^2_{ik} D_j\right].
\end{aligned}$$

Similarly, we obtain

$$\partial^2_{ik} G(\pi^g) = B\left[\frac{1}{p_l - c_l}\partial_i \frac{D_k}{D_l} + \partial^2_{ik} \log D_l\right] + C\left[\partial_i D_k + (p_l - c_l)\partial^2_{ik} D_l\right].$$

Plugging those expressions into the above condition on cross-partial derivatives and using the fact that $\partial_k D_i = \partial_i D_k$ yields:

[14] Although Monderer and Shapley stated their theorem for uni-dimensional action sets, it is straightforward to extend it to multi-dimensional action sets.

$$B\left[\frac{1}{p_j - c_j}\partial_k \frac{D_i}{D_j} + \partial^2_{ik}\log D_j - \frac{1}{p_l - c_l}\partial_i \frac{D_k}{D_l} - \partial^2_{ik}\log D_l\right] + C\left[(p_j - c_j)\partial^2_{ik}D_j - (p_l - c_l)\partial^2_{ik}D_l\right] = 0. \quad (18)$$

As condition (18) must hold on an open set of costs c_j and c_l, we can differentiate it twice with respect to c_j and c_l to obtain $B\partial_k D_i/D_j = 0$ and $B\partial_i D_k/D_l = 0$. Hence $\partial_k D_i/D_j = 0$ and $\partial_i D_k/D_l = 0$ if $B \neq 0$. Moreover, regardless of whether $B \neq 0$, condition (18) reduces to

$$B\partial^2_{ik}\log\frac{D_j}{D_l} + C\left[(p_j - c_j)\partial^2_{ik}D_j - (p_l - c_l)\partial^2_{ik}D_l\right] = 0.$$

As this condition must again hold on an open set of costs c_j and c_l, we can differentiate it once with respect to c_j and c_l to obtain $C\partial^2_{ik}D_j = 0$ and $C\partial^2_{ik}D_l$, which implies that $\partial^2_{ik}D_j = 0$ and $\partial^2_{ik}D_l = 0$ if $C \neq 0$. Moreover, regardless of whether $C \neq 0$, the condition reduces to $B\partial^2_{ik}\log(D_j/D_l) = 0$. Hence, $\partial^2_{ik}\log(D_j/D_l) = 0$ if $B \neq 0$. □

B.4 Proof of Lemma 4

To prove Lemma 4, we split it into two technical lemmas. We begin by integrating the system of partial differential equations in the second part of Lemma 3:

Lemma F. *Suppose that $\partial^2_{ik}D_j = 0$ for every $i, j, k \in \mathcal{N}$ with $k \neq i, j$. Then, the demand system D takes the generalized linear form of equation* (3).

Proof. Fix some j in $\mathcal{N}$. As $\partial_k(\partial_j D_j) = 0$ for every $k \neq j$, we have that $\partial_j D_j$ is independent of p_{-j}. Therefore, there exist functions ϕ_j and ψ_j such that $D_j(p) = \phi_j(p_j) + \psi_j(p_{-j})$ for every $p \in \mathcal{Q}$. Moreover, for every $i, k \neq j$, we have that $\partial^2_{ik}\psi_j(p_{-j}) = \partial^2_{ik}D_j = \partial_{ij}D_k = 0$, where we have used Slutsky symmetry to obtain the second equality. It follows that, for every $i \neq j$, $\partial_i\psi_j$ is equal to some constant α_{ji}. Hence, $\partial_i\left(\psi_j(p_{-j}) - \sum_{j\neq i}\alpha_{ji}p_j\right) = 0$, and so $\psi_j(p_{-j}) = \beta_j + \sum_{j\neq i}\alpha_{ji}p_j$ for some constant of integration β_j. Setting $h'_j(p_j) = -\phi_j(p_j) - \beta_j$ for every j, we obtain the generalized linear form of equation (3). The fact that $\alpha_{ij} = \alpha_{ji}$ follows immediately by Slutsky symmetry. □

Next, we turn to the system of partial differential equations in the first part of Lemma 3:

Lemma G. *Suppose that, for every $i, j, k \in \mathcal{N}$ such that $k \neq i, j$, $\partial_k(D_i/D_j) = 0$, and, for every $i, j \in \mathcal{N}$, $\partial^2_{ij}\log(D_i/D_j) = 0$. Then, the demand system D takes the IIA form of equation* (2).

Proof. Suppose first that $|\mathcal{N}| \geq 3$. Then, the result follows from Proposition 1 in Anderson, Erkal, and Piccinin (2020), the proof of which we replicate here. We have that, for every $p \in \mathcal{Q}$ and every $i, j, k \in \mathcal{N}$ such that $k \neq i, j$, $\partial_k(\partial_i V(p)/\partial_j V(p)) = 0$. Thus, using terminology introduced by Goldman and Uzawa (1964), the function $-V$ is strongly separable with respect to the partition $\{\{n\}\}_{n\in\mathcal{N}}$. Moreover, that function is $\mathcal{C}^3$ on $\mathcal{Q}$, its level sets are connected surfaces, and its partial derivatives are strictly positive everywhere on $\mathcal{Q}$. Theorem 1 in Goldman and Uzawa (1964) then implies that $-V$ takes the form[15]

$$-V(p) = -\Psi\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right).$$

Suppose instead that $|\mathcal{N}| = 2$, and write $\mathcal{N} = \{1, 2\}$. As $\partial^2_{12} \log(D_1/D_2) = 0$, there exist functions ϕ_1 and ϕ_2 such that

$$\log \frac{D_1(p)}{D_2(p)} = \phi_1(p_1) - \phi_2(p_2)$$

for every $p \in \mathcal{Q}$. Taking exponentials, this implies that

$$\frac{D_1(p)}{D_2(p)} = \frac{\mathrm{e}^{\phi_1(p_1)}}{\mathrm{e}^{\phi_2(p_2)}}.$$

For $i = 1, 2$, let h_i be an anti-derivative of e^{ϕ_i}, so that

$$\frac{\partial_1 V(p)}{\partial_2 V(p)} = \frac{h_1'(p_1)}{h_2'(p_2)},$$

which means that there exists a function λ such that

$$\frac{\partial_1 V(p)}{h_1'(p_1)} = \lambda(p) = \frac{\partial_2 V(p)}{h_2'(p_2)}.$$

By Lemma 1 in Goldman and Uzawa (1964), there thus exists a function Ψ such that

$$V(p) = \Psi\left(h_1(p_1) + h_2(p_2)\right). \qquad \square$$

C Proof of Proposition 2

We use the following notation throughout this section: for every $p \in \mathcal{Q}$, $c \in \mathbb{R}^{\mathcal{N}}_{++}$, and $f \in \mathcal{F}$,

$$\pi^f(p, c) \equiv \sum_{j\in f}(p_j - c_j)D_j(p).$$

[15] Although Goldman and Uzawa stated their results for utility functions defined on the entire non-negative orthant, their proofs continue to go through for utility functions defined over a convex subset of that orthant.

C.1 Proof that (b) implies (a)

In this subsection, we show that (b) implies (a) and that the logarithm (resp. the identity function) is an admissible transformation function for the demand system. If (b)-(ii) holds, then this follows immediately from the fact that the payoff gradient is equal to the gradient of the potential function defined in equation (8).

Suppose instead that (b)-(i) holds. To prove that the logarithm is an admissible transformation function for the demand system, all we need to do is show that Monderer and Shapley (1996b)'s necessary and sufficient condition holds for the pricing game with transformed payoffs at every $p \in Q$ and $c \in \prod_{j \in \mathcal{N}}(0, p_j)$. That is, we need to show that, for every $f, g \in \mathcal{F}$ such that $f \neq g$, for every $i \in f$ and $k \in g$,

$$\frac{\partial^2}{\partial p_i \partial p_k} \log \pi^f(p, c) = \frac{\partial^2}{\partial p_i \partial p_k} \log \pi^g(p, c). \tag{19}$$

We have that

$$\partial^2_{ik} \log \pi^f = \partial^2_{ik} \left(\log D_i + \log \left[\sum_{j \in f} (p_j - c_j) \frac{D_j}{D_i} \right] \right) = \partial^2_{ik} \log D_i,$$

where the second equality follows as $\partial_k (D_i / D_j) = 0$ for every $j \in f$. Similarly, $\partial^2_{ik} \log \pi^g = \partial^2_{ik} \log D_k$, so that condition (19) reduces to $\partial^2_{ik} \log D_i = \partial^2_{ik} \log D_k$, which holds by assumption.

C.2 Proof that (a) implies (b)

We split this part into a series of technical lemmas. We begin by stating the analogue of Lemma 1:

Lemma H. *Let $f, g \in \mathcal{F}$ with $f \neq g$, $i \in f$ and $j \in g$. There exists a function $\kappa(\cdot)$ such that for every $\pi \in (0, \overline{\pi}_i)$ and $p \in Q_i(\pi)$,*

$$\partial_j D_i \left(1 + \pi \frac{\partial_i D_i}{D_i^2} \right) \varphi'(\pi) + \left(\frac{\partial^2_{ij} D_i}{D_i} - \frac{\partial_i D_i \partial_j D_i}{D_i^2} \right) \varphi(\pi) = \kappa(p), \tag{20}$$

where the function D_i and its partial derivatives are all evaluated at p.

Proof. Let $p \in \mathcal{Q}$ and c be a marginal cost vector such that $c_k = p_k$ for every $k \in f \cup g \setminus \{i, j\}$ and $c_k \in (0, p_k)$ for every other product k. Theorem 4.5 in Monderer and Shapley (1996b) implies that $\partial^2_{ij} G^f(\pi^f(p, c)) = \partial^2_{ij} G^g(\pi^g(p, c))$. As $p_k = c_k$ for every $k \in f \cup g \setminus \{i, j\}$, this is equivalent to $\partial^2_{ij} G(\pi_i(p, c_i)) = \partial^2_{ij} G(\pi_j(p, c_j))$, i.e., condition (14) in the proof of Lemma 1 holds. The rest of the proof of that lemma can then be replicated word for word to obtain the result. □

Next, we state the analogue of Lemma 2:

Lemma I. *There exist constants $\hat{\pi} > 0$, A, B, and C such that $B + C\pi > 0$ and $G(\pi) = A + B\log\pi + C\pi$ for every $\pi \in (0, \hat{\pi})$.*

Proof. Given Lemma H, the proof of Lemma 2 can be replicated to obtain the result. □

Next, we state the analogue of Lemma 3:

Lemma J. *Let $f, g \in \mathcal{F}$, $f \neq g$, and $(i, j, k) \in f \times f \times g$. If $B \neq 0$ in Lemma I, then $\partial_k(D_i/D_j) = 0$ and $\partial^2_{ik} \log(D_i/D_k) = 0$. If $C \neq 0$ in Lemma I, then $\partial^2_{ik} D_j = 0$.*

Proof. Given Lemma I, we can proceed as in the proof of Lemma 3. □

To integrate the system of partial differential equations in the second part of Lemma J, we require the following technical lemma:

Lemma K. *Let $n \geq 2$ and Π be a partition of $\{1, \ldots, n\}$ containing at least two elements. Let $F : X \longrightarrow \mathbb{R}$ be $|\Pi|+1$ times continuously differentiable over its open and convex domain $X \subseteq R^n$. Suppose that, for every $\pi \in \Pi$, for every $i, j \in \pi$, $\partial^2_{ij} F = 0$. Then, F takes the form*

$$F(x) = \alpha + \sum_{\Pi' \subseteq \Pi} \sum_{\iota \in \prod_{\pi \in \Pi'} \pi} \alpha(\iota) \prod_{\pi \in \Pi'} x_{\iota(\pi)}.$$

Proof. We prove the result by induction on $|\Pi|$. If $|\Pi| = 1$, then $\partial^2_{ij} F = 0$ for every $1 \leq i, j \leq n$, and so $\partial_i F$ is equal to some constant α_i for every i. It follows that $F(x) = \alpha + \sum_{i=1}^n \alpha_i x_i$ for some α, establishing the property for $|\Pi| = 1$.

Next, suppose that $|\Pi| > 1$ and that the property holds for $|\Pi| - 1$. Let $\pi_0 \in \Pi$. We have that $\partial^2_{ij} F = 0$ for every $i, j \in \pi_0$, implying that, for every $i \in \pi_0$, $\partial_i F(x)$ is equal to some $\mathcal{C}^{|\Pi|}$ function $\beta_i(x^{-\pi_0})$ of the subvector $x^{-\pi_0} = (x_j)_{\substack{1 \leq j \leq n \\ j \notin \pi_0}}$. Observe that Y, the domain of β_i, is open and convex.[16] Thus, for every $i \in \pi_0$, we have that $\partial_i \left(F(x) - \sum_{j \in \pi_0} \beta_j(x^{-\pi_0}) x_j \right) = 0$, which implies the existence of a $\mathcal{C}^{|\Pi|}$ function $\beta_0(x^{-\pi_0})$ such that

$$F(x) = \beta_0(x^{-\pi_0}) + \sum_{j \in \pi_0} \beta_j(x^{-\pi_0}) x_j. \tag{21}$$

Let $\pi \in \Pi \setminus \{\pi_0\}$. For every $k, \ell \in \pi$, we have that

$$\partial^2_{k\ell} F = \partial^2_{k\ell} \beta_0(x^{-\pi_0}) + \sum_{j \in \pi_0} \partial^2_{k\ell} \beta_j(x^{-\pi_0}) x_j = 0.$$

[16] Convexity is immediate. To see why Y is open, note that $Y = \bigcup_{z \in \mathbb{R}^{\pi_0}} \{y : \ (z, y) \in X\}$ and each of the sets in the union is open.

For every $x^{-\pi_0} \in Y$, the above condition has to hold on the open set $\{x^{\pi_0} = (x_j)_{j\in\pi_0} : (x^{\pi_0}, x^{-\pi_0}) \in X\}$. Hence, $\partial^2_{k\ell}\beta_0 = 0$ and $\partial^2_{k\ell}\beta_j = 0$ for every $j \in \pi$. We can therefore apply the induction hypothesis to each of the β functions, $\mathcal{C}^{|\Pi|}$ over the open and convex domain Y, with partition $\Pi \setminus \{\pi_0\}$, to obtain that, for every $j \in \pi_0 \cup \{0\}$, β_j takes the form

$$\beta_j(y) = \alpha_j + \sum_{\Pi' \subseteq \Pi\setminus\{\pi_0\}} \sum_{\iota \in \prod_{\pi\in\Pi'} \pi} \alpha(\iota) \prod_{\pi\in\Pi'} y_{\iota(\pi)}.$$

Combining this with equation (21) proves the lemma. $\square$

Armed with Lemma K, we integrate the system of partial differential equations in the second part of Lemma J:

Lemma L. *Assume that, for every $f, g \in \mathcal{F}$ with $f \neq g$, and $(i, j, k) \in f \times f \times g$, $\partial^2_{ik}D_j = 0$. Then, D takes the following form: For any $f \in \mathcal{F}$ and $i \in f$,*

$$D_i(p) = -\partial_i\psi^f(p^f) + \sum_{\substack{\mathcal{F}'\subseteq\mathcal{F}:\\ f\in\mathcal{F}'}} \sum_{\substack{\iota\in\prod_{g\in\mathcal{F}'} g:\\ \iota(f)=i}} \alpha(\iota) \prod_{\substack{g\in\mathcal{F}':\\ g\neq f}} p_{\iota(g)}.$$

Proof. Let $f \in \mathcal{F}$ and $i \in f$. As $\partial^2_{jk}D_i = 0$ for every $j \in f$ and $k \notin f$, we have that, for every $j \in f$, $\partial_j D_i = \phi^j(p^f)$, for some function ϕ^j of firm f's price vector p^f. As the functions $(\phi^j(p^f))_{j\in f}$ satisfy $\partial_\ell\phi^j = \partial_{j\ell}D_i = \partial_j\phi^\ell$ for every $j, \ell \in f$, the Poincaré lemma implies the existence of a function $d_i(p^f)$ such that $\partial_j d_i(p^f) = \phi^j(p^f)$ for every $j \in f$. Hence, $\partial_j(D_i(p) - d_i(p^f)) = 0$ for every $j \in f$, which implies that D_i can be written as $D_i(p) = d_i(p^f) + \delta_i(p^{-f})$. Moreover, as $\partial_j d_i = \partial_j D_i = \partial_i D_j = \partial_i d_j$ for every $i, j \in f$, the Poincaré lemma implies the existence of a function $\psi^f(p^f)$ such that $d_i(p^f) = -\partial_i\psi^f(p^f)$ for every $i \in f$.

For every $g \in \mathcal{F} \setminus \{f\}$ and $k, \ell \in g$, we have that

$$\partial^2_{k\ell}\delta_i(p^{-f}) = \partial^2_{k\ell}D_i = \partial^2_{ki}D_\ell = 0,$$

where the second equality follows by Slutsky symmetry. We can therefore apply Lemma K to the $\mathcal{C}^{|\mathcal{F}|}$ function δ_i (with partition $\mathcal{F} \setminus \{f\}$): this function must take the form

$$\delta_i(p^{-f}) = \alpha_i + \sum_{\mathcal{F}'\subseteq\mathcal{F}\setminus\{f\}} \sum_{\iota\in\prod_{g\in\mathcal{F}'} g} \alpha_i(\iota) \prod_{g\in\mathcal{F}'} p_{\iota(g)}.$$

Rewriting, this means that for some weights $\tilde{\alpha}_i(\iota)$,

$$\delta_i = \partial_i \sum_{\mathcal{F}'\subseteq\mathcal{F}} \sum_{\iota\in\prod_{g\in\mathcal{F}'} g} \tilde{\alpha}_i(\iota) \prod_{g\in\mathcal{F}'} p_{\iota(g)}.$$

Moreover, as, for every ι, the product $\prod_{g\in\mathcal{F}'} p_{\iota(g)}$ contains at most one of firm f's prices, we can choose the weights $\tilde{\alpha}_i(\iota)$ such that, for every $i,j\in f$ and every ι, $\tilde{\alpha}_i(\iota)=\tilde{\alpha}_j(\iota)\equiv\alpha^f(\iota)$, yielding

$$\delta_i=\partial_i\sum_{\mathcal{F}'\subseteq\mathcal{F}}\ \sum_{\iota\in\prod_{g\in\mathcal{F}'}g}\alpha^f(\iota)\prod_{g\in\mathcal{F}'}p_{\iota(g)}.$$

Let $f'\in\mathcal{F}\setminus\{f\}$ and $(i,k)\in f\times f'$. By Slutsky symmetry, we have that $\partial_k\delta_i=\partial_i\delta_k$, implying that, for every $p\in\mathcal{Q}$,

$$\sum_{\substack{\mathcal{F}'\subseteq\mathcal{F}:\\ f,f'\in\mathcal{F}'}}\ \sum_{\substack{\iota\in\prod_{g\in\mathcal{F}'}g:\\ \iota(f)=i\text{ and }\iota(f')=k}}\left[\alpha^f(\iota)-\alpha^{f'}(\iota)\right]\prod_{\substack{g\in\mathcal{F}':\\ g\neq f,f'}}p_{\iota(g)}=0.$$

As the above condition has to hold on an open set of prices, it must be that $\alpha^f(\iota)=\alpha^{f'}(\iota)$ for every $\mathcal{F}'$ such that $f,f'\in\mathcal{F}'$ and $\iota\in\prod_{g\in\mathcal{F}'}g$ such that $\iota(f)=i$ and $\iota(f')=k$. Hence, there exists a function $\alpha(\cdot)$, defined over $\bigcup_{\mathcal{F}'\subseteq\mathcal{F}}\prod_{g\in\mathcal{F}'}g$ such that, for every $f\in\mathcal{F}$, $\mathcal{F}'\subseteq\mathcal{F}$ such that $f\in\mathcal{F}'$, and $\iota\in\prod_{g\in\mathcal{F}'}g$, $\alpha^f(\iota)=\alpha(\iota)$. In fact, since the weight $\alpha^f(\iota)$ is irrelevant for firm f's demand whenever $\iota\in\prod_{g\in\mathcal{F}'}g$ with $f\notin\mathcal{F}'$, we can write

$$\delta_i=\partial_i\sum_{\mathcal{F}'\subseteq\mathcal{F}}\ \sum_{\iota\in\prod_{g\in\mathcal{F}'}g}\alpha(\iota)\prod_{g\in\mathcal{F}'}p_{\iota(g)}=\sum_{\substack{\mathcal{F}'\subseteq\mathcal{F}:\\ f\in\mathcal{F}'}}\ \sum_{\substack{\iota\in\prod_{g\in\mathcal{F}'}g:\\ \iota(f)=i}}\alpha(\iota)\prod_{\substack{g\in\mathcal{F}':\\ g\neq f}}p_{\iota(g)}$$

for every $i\in f$. □

D Proof of Proposition 3

Proof. Let $\mathcal{F}=\{f_1,f_2\}$. The "if" part of the proposition follows immediately from the fact that the gradient of the potential function in equation (9) coincides with the log-payoff gradient.

Conversely, suppose that $(D,\mathcal{F})$ admits a log-potential. By Lemma I, for every $f,g\in\mathcal{F}$ with $f\neq g$, $i,j\in f$, and $k\in g$, we have that $\partial_k D_i/D_j=0$ and $\partial^2_{ik}\log D_i/D_k=0$. As $\partial_k(\partial_i V/\partial_j V)=0$ for every $f,g\in\mathcal{F}$ with $f\neq g$, $i,j\in f$, and $k\in g$, the function V is weakly separable with respect to the partition $\mathcal{F}$. Hence, by Theorem 2 in Goldman and Uzawa (1964), there exist functions Λ and ϕ^f such that, for every p, we have:

$$V(p)=\Lambda\left(\phi^{f_1}\left(p^{f_1}\right),\phi^{f_2}\left(p^{f_2}\right)\right).$$

Hence, for every $\iota\in\{1,2\}$ and $i\in f_\iota$, $D_i(p)=-\partial_i\phi^{f_\iota}\left(p^{f_\iota}\right)\partial_\iota\Lambda\left(\phi^{f_1}\left(p^{f_1}\right),\phi^{f_2}\left(p^{f_2}\right)\right)$.

Let $i\in f_1$ and $k\in f_2$. We have:

$$0=\partial^2_{ik}\log\frac{D_i}{D_k}=\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_k}\log\left[\frac{\partial_i\phi^{f_1}\left(p^{f_1}\right)}{\partial_k\phi^{f_2}\left(p^{f_2}\right)}\frac{\partial_1\Lambda\left(\phi^{f_1}\left(p^{f_1}\right),\phi^{f_2}\left(p^{f_2}\right)\right)}{\partial_2\Lambda\left(\phi^{f_1}\left(p^{f_1}\right),\phi^{f_2}\left(p^{f_2}\right)\right)}\right]$$

$$= \frac{\partial \phi^{f_1}\left(p^{f_1}\right)}{\partial p_i} \frac{\partial}{\partial \phi^{f_1}} \frac{\partial \phi^{f_2}\left(p^{f_2}\right)}{\partial p_k} \frac{\partial}{\partial \phi^{f_2}} \log \frac{\partial_1 \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)}{\partial_2 \Lambda\left(\phi^{f_1}\left(p^{f_1}\right), \phi^{f_2}\left(p^{f_2}\right)\right)}$$

$$= \partial_i \phi^{f_1}\left(p^{f_1}\right) \partial_k \phi^{f_2}\left(p^{f_2}\right) \partial^2_{12} \log \frac{\partial_1 \Lambda(x_1, x_2)}{\partial_2 \Lambda(x_1, x_2)}\bigg|_{x_1 = \phi^{f_1}\left(p^{f_1}\right),\ x_2 = \phi^{f_2}\left(p^{f_2}\right)}.$$

As $\partial_i \phi^{f_1}\left(p^{f_1}\right) \neq 0$ and $\partial_k \phi^{f_2}\left(p^{f_2}\right) \neq 0$ for every $p \in \mathcal{Q}$, it follows that

$$\partial^2_{12} \log \frac{\partial_1 \Lambda(x_1, x_2)}{\partial_2 \Lambda(x_1, x_2)}$$

for every (x_1, x_2) in the domain of Λ.

We integrated that same partial differential equation in the second half of the proof of Lemma 4: the solutions take the form $\Lambda(x_1, x_2) = \Psi(h_1(x_1) + h_2(x_2))$ for some functions Ψ, h_1, and h_2. It follows that

$$V(p) = \Psi\left[h_1\left(\phi^{f_1}\left(p^{f_1}\right)\right) + h_2\left(\phi^{f_2}\left(p^{f_2}\right)\right)\right]$$

Defining $\psi^{f_\iota} \equiv h_\iota \circ \phi^{f_\iota}$ and using Roy's identity proves the proposition. □

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